

THE EXPANDED MIXED FINITE ELEMENT METHOD FOR GENERALIZED FORCHHEIMER FLOWS IN POROUS MEDIA*

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Abstract. We study the expanded mixed finite element method applied to degenerate parabolic equations with the Dirichlet boundary condition. The equation is considered a prototype of the nonlinear Forchheimer equation, a inverted to the nonlinear Darcy equation with permeability coefficient depending on pressure gradient, for slightly compressible fluid flow in porous media. The bounds for the solutions are established. In both continuous and discrete time procedures, utilizing the monotonicity properties of Forchheimer equation and boundedness of solutions we prove the optimal error estimates in L^2 -norm for solution. The error bounds are established for the solution and divergence of the vector variable in Lebesgue norms and Sobolev norms under some additional regularity assumptions. A numerical example using the lowest order Raviart-Thomas (RT_0) mixed element are provided agreement with our theoretical analysis.

Key words. Expanded mixed finite element, nonlinear degenerate parabolic equations, generalized Forchheimer equations, error estimates.

AMS subject classifications. 65M12, 65M15, 65M60, 35Q35, 76S05.

1. Introduction. The paper is dedicated to the analysis of mixed finite element approximation of the solution of the system modeling the flows of compressible fluid in porous media subjected to generalized Forchheimer law. Forchheimer type flow belongs to the so called post-Darcy class of flows and are designed to model high velocity filtration in porous media when inertial and friction terms cannot be ignored. In recent years, this phenomena generated a lot of interest in the research community in many areas of engineering, environmental and groundwater hydrology and in medicine.

Accurate description of fluid flow behavior in porous media is essential to successful forecasting and project design in reservoir engineering. Most of the analyses of the flow in porous media are based on Darcy law, which describes a linear relationship between the pressure gradient and the fluid velocity. However, when the fluid in porous media flows at very high or very low velocity, the Darcy law is no longer valid. Reservoir engineers often divide flows in the media into three main categories with respect to Darcy law (cf. [31]): fast flows near the well and fracture (post-Darcy), linear non-fast/non-slow flows described by Darcy equation in the main domain between near well zone and “far a way” region, and on the periphery of the media, where the impact of the well is small. A nonlinear relationship between velocity and gradient of pressure is introduced adding the higher order term of velocity in the Darcy equation. In this research we concentrate on the first type of flow, when deviation from linear Darcy is associated with high velocity field.

Engineers commonly use Forchheimer equation to take into account inertial phenomena. In early 1900s, Forchheimer proposed three models for nonlinear flows, the so-called two-term, three-term and power laws (cf. [11]) to match experimental observations. There is a significant number of papers studying these equations and their variations the Brinkman-Forchheimer equations for incompressible fluids

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(cf. [6, 7, 12, 13, 30, 32]). Recently the authors in [3, 15, 16, 17, 18, 19] proposed and studied generalized Forchheimer equations for slightly compressible fluids in porous media. These works focus on theory of existence, stability and qualitative property of solutions within framework of non-linear parabolic equation with coefficient degenerating as gradient of pressure converges to infinity. To apply developed models and method to practical problem, it is important to investigate properties and convergence of the approximate numerical solutions of corresponding degenerate parabolic equations.

The popular numerical methods for modeling flow in porous media are mixed finite element approximations (cf. [9, 14, 22, 28, 29]) and block-centered finite difference method (cf. [34]). These methods are widely used because they inherit conservation properties and because they produce accurate flux even for highly homogeneous media with large jumps of conductivity (permeability) tensor (cf. [10]). Arbogast, Wheeler and Zhang in [2] analyzed mixed finite element approximations of degenerate parabolic equation. However according to Arbogast, Wheeler and Yotov in [1], the standard mixed finite element method is not suitable for problems with degenerating tensor coefficients as the tensor needs to be inverted.

The proposed approach reduces original Forchheimer type equation to generalized Darcy equation with conductivity tensor K degenerating as gradient of the pressure convergence to infinity. At the same time, the standard mixed variational formulation requires inverting K . Woodward and Dawson in [36] studied expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media. Compared with the standard mixed finite element method, the expanded mixed finite element method introduces three variables: unknown scalar function, its gradient, and a flux. In their analysis, the Kirchhoff transformations are used to move the nonlinearity from K term to the gradient and thus analysis of the equations is simplified. This transformation is not applicable to our system (3.1).

In this paper, we will employ techniques developed in [15] and the expanded mixed finite element method presented in [1]. The combination of these techniques enables us to utilize both the special structure of the equation and the advantages of the expanded mixed finite element method, for instance, providing certain implementational advantages over the standard mixed method, in particular for lowest order Raviart-Thomas (RT) space on the rectangular grids, in obtaining the optimal order error estimates for the solution in several norms of interest.

The outline of this paper is as follows: In section 2, we introduce the generalized formulation of the Forchheimers laws for slightly compressible fluids, recall relevant results in [3, 15] and preliminary results. In section 3, we present the expanded mixed formulation and standard results for mixed finite element approximations. An implicit backward-difference time discretization of the semidiscrete scheme is proposed to solve the system (3.2). In section 4, we derive many bounds for solutions to (3.3) and (3.7) in Lebesgue norms in term of boundary data and the initial data. In section 5, we establish error estimates in L^2 -norms, L^∞ -norm and H^{-1} -norm for pressure. Then the error estimates for gradient of pressure and flux variable are also derived under reasonable assumptions on the regularity of solutions. Also the error analysis for fully discrete version is obtained in suitable norm for the three relevant variables. In section 6, we give a numerical example using the lowest Raviart-Thomas mixed finite element to support our theoretical analysis.

2. Preliminaries and auxiliaries. In this paper, we consider a fluid in a porous medium in a bounded domain $\Omega \subset \mathbb{R}^d, d \geq 2$. Its boundary $\Gamma = \partial\Omega$ belongs to C^2 .

Let $x \in \mathbb{R}^d, 0 < T < \infty, t \in (0, T]$ be the spatial and time variable. The fluid flow has velocity $\mathbf{u}(x, t) \in \mathbb{R}^d$, pressure $p(x, t) \in \mathbb{R}$.

A general Forchheimer equation which is studied in [3] has the form

$$g(|\mathbf{u}|)\mathbf{u} = -\nabla p, \quad (2.1)$$

where $g(s) = a_0 + a_1 s^{\alpha_1} + \dots + a_N s^{\alpha_N}$, $s \geq 0$, $N \geq 1$. $0 < \alpha_1 < \dots < \alpha_N$ are fixed numbers, the coefficients a_0, \dots, a_N are non-negative numbers with $a_0 > 0, a_N > 0$. In particular when $g(s) = \alpha, \alpha + \beta s, \alpha + \beta s + \gamma s^2 + \gamma_m s^{m-1}$, where $\alpha, \beta, \gamma, m, \gamma_m$ are empirical constants, we have Darcy's law, Forheimer's two term, three term and power law, respectively.

The monotonicity of the nonlinear term and the non-degeneracy of the Darcy's parts in (2.1) enable us to write \mathbf{u} implicit in terms of ∇p :

$$\mathbf{u} = -K(|\nabla p|)\nabla p. \quad (2.2)$$

Here the function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$K(\xi) = \frac{1}{g(s(\xi))} \text{ where } s = s(\xi) \geq 0 \text{ satisfies } sg(s) = \xi, \text{ for } \xi \geq 0. \quad (2.3)$$

The state equation, which relates the density $\rho(x, t) > 0$ with pressure $p(x, t)$, for slightly compressible fluids is

$$\frac{d\rho}{dp} = \kappa^{-1} \rho \text{ or } \rho(p) = \rho_0 \exp\left(\frac{p - p_0}{\kappa}\right), \quad \kappa > 0. \quad (2.4)$$

Other equations governing the fluid's motion are the equation of continuity:

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

which yields

$$\frac{d\rho}{dp} \frac{dp}{dt} + \rho \nabla \cdot \mathbf{u} + \frac{d\rho}{dp} \mathbf{u} \cdot \nabla p = 0. \quad (2.5)$$

Combining (2.4) and (2.5), we obtain

$$\frac{dp}{dt} + \kappa \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p = 0. \quad (2.6)$$

Since the constant κ is very large for most slightly compressible fluids in porous media [26], in most of the practical application the third term on the left-hand side of (2.6) is neglected. This results in the following reduced equation

$$\frac{dp}{dt} + \kappa \nabla \cdot \mathbf{u} = 0. \quad (2.7)$$

By rescaling the time variable, hereafter we assume that $\kappa = 1$. From (2.2) and (2.7) we have the system

$$\begin{cases} \mathbf{u} + K(|\nabla p|)\nabla p = 0, \\ p_t + \nabla \cdot \mathbf{u} = 0. \end{cases} \quad (2.8)$$

System (2.8) gives a scalar equation for pressure:

$$p_t - \nabla \cdot (K(|\nabla p|)\nabla p) = 0. \quad (2.9)$$

The function $K(\xi)$ has the important properties (cf. [3, 15]):

- (i) $K : [0, \infty) \rightarrow (0, a_0^{-1}]$ and it decreases in ξ ,
- (ii) Type of degeneracy

$$\frac{c_1}{(1+\xi)^a} \leq K(\xi) \leq \frac{c_2}{(1+\xi)^a}, \quad (2.10)$$

- (iii) For all $n \geq 1$,

$$c_3(\xi^{n-a} - 1) \leq K(\xi)\xi^n \leq c_2\xi^{n-a}, \quad (2.11)$$

- (iv) Relation with its derivative

$$-aK(\xi) \leq K'(\xi)\xi \leq 0, \quad (2.12)$$

where c_1, c_2, c_3 are positive constants depending on Ω and g . The constant $a \in (0, 1)$ is defined by

$$a = \frac{\alpha_N}{\alpha_N + 1} = \frac{\deg(g)}{\deg(g) + 1}. \quad (2.13)$$

The following function is crucial in our estimates. We define

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s})dx, \text{ for } \xi \geq 0. \quad (2.14)$$

The function $H(\xi)$ can compare with ξ and $K(\xi)$ by

$$K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2, \quad (2.15)$$

as a consequence of (2.11) and (2.15) we have,

$$C(\xi^{2-a} - 1) \leq H(\xi) \leq 2C\xi^{2-a}. \quad (2.16)$$

For the monotonicity and continuity of the differential operator in (2.8) we have the following results.

PROPOSITION 2.1 (cf. [15]). *One has*

- (i) For all $y, y' \in \mathbb{R}^d$,

$$(K(|y'|)y' - K(|y|)y) \cdot (y' - y) \geq (1-a)K(\max\{|y|, |y'|\})|y' - y|^2. \quad (2.17)$$

- (ii) For the vector functions $\mathbf{s}_1, \mathbf{s}_2$, there is a positive constant C such that

$$\int_{\Omega} (K(|\mathbf{s}_1|)\mathbf{s}_1 - K(|\mathbf{s}_2|)\mathbf{s}_2) \cdot (\mathbf{s}_1 - \mathbf{s}_2) dx \geq C\omega \|\mathbf{s}_1 - \mathbf{s}_2\|_{L^\beta(\Omega)}^2, \quad (2.18)$$

where

$$\omega = (1 + \max\{\|\mathbf{s}_1\|_{L^\beta(\Omega)}, \|\mathbf{s}_2\|_{L^\beta(\Omega)}\})^{-a}. \quad (2.19)$$

PROPOSITION 2.2. *For all $y, y' \in \mathbb{R}^d$ we have*

$$|K(|y'|)y' - K(|y|)y| \leq \sqrt{2(a^2 + 1)}a_0^{-1}|y' - y|. \quad (2.20)$$

Proof. Case 1: The origin does not belong to the segment connect y' and y . Let $\ell(t) = ty' + (1-t)y, t \in [0, 1]$. Define $h(t) = K(|\ell(t)|)\ell(t)$ for $t \in [0, 1]$. By the mean value theorem, there is $t_0 \in [0, 1]$ with $\ell(t_0) \neq 0$, such that

$$\begin{aligned} |K(|y'|)y' - K(|y|)y|^2 &= |h(1) - h(0)|^2 = |h'(t_0)|^2 \\ &= \left| K'(|\ell(t_0)|) \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|} \ell(t_0) + K(|\ell(t_0)|)\ell'(t_0) \right|^2. \end{aligned}$$

Using (2.12) and Minkowski's inequality we obtain

$$|K(|y'|)y' - K(|y|)y|^2 \leq 2|K(|\ell(t_0)|)|^2 \left(a^2 \left| \frac{\ell(t_0) \cdot \ell'(t_0)}{|\ell(t_0)|^2} \ell(t_0) \right|^2 + |\ell'(t_0)|^2 \right).$$

The (2.20) follows by the boundedness of $K(\cdot) \leq a_0^{-1}$.

Case 2: The origin belongs to the segment connect y', y . We replace y' by some $y_\varepsilon \neq 0$ so that $0 \notin [y_\varepsilon, y]$ and $y_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Apply the above inequality for y and y_ε , then let $\varepsilon \rightarrow 0$. \square

Notations. Let $L^2(\Omega)$ be the set of square integrable functions on Ω and $(L^2(\Omega))^d$ the space of d -dimensional vectors which have all components in $L^2(\Omega)$.

We denote (\cdot, \cdot) the inner product in either $L^2(\Omega)$ or $(L^2(\Omega))^d$ that is

$$(\xi, \eta) = \int_{\Omega} \xi \eta dx \quad \text{or} \quad (\xi, \eta) = \int_{\Omega} \xi \cdot \eta dx.$$

The notation $\|\cdot\|$ will means scalar norm $\|\cdot\|_{L^2(\Omega)}$ or vector norm $\|\cdot\|_{(L^2(\Omega))^d}$.

For $1 \leq q \leq +\infty$ and m any nonnegative integer, let

$$W^{m,q}(\Omega) = \{f \in L^q(\Omega), D^\alpha f \in L^q(\Omega), |\alpha| \leq m\}$$

denote a Sobolev space endowed with the norm

$$\|f\|_{m,q} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^q(\Omega)}^q \right)^{\frac{1}{q}}.$$

Define $H^m(\Omega) = W^{m,2}(\Omega)$ with the norm $\|\cdot\|_m = \|\cdot\|_{m,2}$.

For functions p, u and vector-functions $\mathbf{v}, \mathbf{s}, \mathbf{u}$ we use short hand notations

$$\|p(t)\| = \|p(\cdot, t)\|_{L^2(\Omega)}, \quad \|\mathbf{u}(t)\| = \|\mathbf{u}(\cdot, t)\|_{L^2(\Omega)}, \quad \|\mathbf{s}(t)\|_{L^\beta(\Omega)} = \|\mathbf{s}(\cdot, t)\|_{L^\beta(\Omega)}$$

and

$$u^0(\cdot) = u(\cdot, 0), \quad \mathbf{v}^0(\cdot) = \mathbf{v}(\cdot, 0).$$

Throughout this paper the constants

$$\beta = 2 - a, \quad \gamma = \frac{a}{2 - a}.$$

The arguments C, C_1 will represent for positive generic constants and their values depend on exponents, coefficients of polynomial g , the spatial dimension d and domain Ω , independent of the initial and boundary data, size of mesh and time step. These constants may be different place by place.

3. The expanded mixed finite element methods. We introduce the new variable $\mathbf{s} = \nabla p$ to (2.8) and study the initial value boundary problem (IVBP):

$$\begin{cases} p_t + \nabla \cdot \mathbf{u} = f, \\ \mathbf{u} + K(|\mathbf{s}|)\mathbf{s} = 0, \\ \mathbf{s} - \nabla p = 0, \end{cases} \quad (3.1)$$

for all $x \in \Omega, t \in (0, T)$, where $f : \Omega \times (0, T) \rightarrow \mathbb{R}, f \in C^1([0, T]; L^\infty(\Omega))$.

The initial and boundary conditions:

$$p(x, 0) = p_0(x) \text{ in } \Omega, \quad p(x, t) = \psi(x, t) \text{ on } \Gamma \times (0, T),$$

we also require at $t = 0$: $p_0(x) = \psi(x, 0)$ on boundary Γ .

To deal with the non-homogeneous boundary condition, we extend the Dirichlet boundary data from boundary Γ to the whole domain Ω (see [15, 20, 25]). Let $\Psi(x, t)$ be a such extension. Let $\bar{p} = p - \Psi$. Then $\bar{p}(x, t) = 0$ on $\Gamma \times (0, T)$. System (3.1) rewrites as

$$\begin{cases} \bar{p}_t + \nabla \cdot \mathbf{u} = -\Psi_t + f, \\ \mathbf{u} + K(|\mathbf{s}|)\mathbf{s} = 0, \\ \mathbf{s} - \nabla \bar{p} = \nabla \Psi, \end{cases} \quad (3.2)$$

for all $(x, t) \in \Omega \times (0, T)$, where $\bar{p}(x, 0) = p_0(x) - \Psi(x, 0) = \bar{p}_0(x)$.

Define $W = L^2(\Omega)$, $\tilde{W} = (L^2(\Omega))^d$, and the Hilbert space

$$V = H(\text{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^d, \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

with the norm defined by $\|\mathbf{v}\|_V^2 = \|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2$.

The variational formulation of (3.2) is defined as the following: Find $(p, \mathbf{u}, \mathbf{s}) : [0, T] \rightarrow W \times V \times \tilde{W}$ such that

$$(\bar{p}_t, w) + (\nabla \cdot \mathbf{u}, w) = (f - \Psi_t, w), \quad \forall w \in W, \quad (3.3a)$$

$$(\mathbf{u}, \mathbf{z}) + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{z}) = 0, \quad \forall \mathbf{z} \in \tilde{W}, \quad (3.3b)$$

$$(\mathbf{s}, \mathbf{v}) + (\bar{p}, \nabla \cdot \mathbf{v}) = (\nabla \Psi, \mathbf{v}), \quad \forall \mathbf{v} \in V \quad (3.3c)$$

with $\bar{p}(x, 0) = \bar{p}_0(x)$.

Semidiscrete method. Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform triangulations of Ω with h being the maximum diameter of the element. Let V_h be the Raviart-Thomas-Nédélec spaces [27, 33] of order $r \geq 0$ or Brezzi-Douglas-Marini spaces [4] of index r over each triangulation \mathcal{T}_h , W_h the space of discontinuous piecewise polynomials of degree r over \mathcal{T}_h , \tilde{W}_h the n -dimensional vector space of discontinuous piecewise polynomials of degree r over \mathcal{T}_h . Let $W_h \times V_h \times \tilde{W}_h$ be the mixed element spaces approximating to $W \times V \times \tilde{W}$.

We use the standard L^2 -projection operator (see [8]) $\pi : W \rightarrow W_h, \pi : \tilde{W} \rightarrow \tilde{W}_h$ satisfying

$$(\pi w, \nabla \cdot \mathbf{v}_h) = (w, \nabla \cdot \mathbf{v}_h)$$

for all $w \in W, \mathbf{v}_h \in V_h$, and

$$(\pi \mathbf{z}, \mathbf{z}_h) = (\mathbf{z}, \mathbf{z}_h)$$

for all $\mathbf{z} \in \tilde{W}$, $\mathbf{z}_h \in \tilde{W}_h$.

Also we use H -div projection $\Pi : V \rightarrow V_h$ defined by

$$(\nabla \cdot \Pi \mathbf{v}, w_h) = (\nabla \cdot \mathbf{v}, w_h)$$

for all $w_h \in W_h$.

These projections have well-known approximation properties as in [5, 21].

(i) $\|\pi w\| \leq \|w\|$ holds for all $w \in L^2(\Omega)$.

(ii) There exist positive constants C_1, C_2 such that

$$\|\pi w - w\|_{L^\alpha(\Omega)} \leq C_1 h^m \|w\|_{m,\alpha} \quad \text{and} \quad \|\pi \mathbf{z} - \mathbf{z}\|_{L^\alpha(\Omega)} \leq C_2 h^m \|\mathbf{z}\|_{m,\alpha}, \quad (3.4)$$

for all $w \in W^{m,\alpha}(\Omega)$, $\mathbf{z} \in (W^{m,\alpha}(\Omega))^d$, $0 \leq m \leq r+1$, $1 \leq \alpha \leq \infty$. Here $\|\cdot\|_{m,\alpha}$ denotes a standard norm in Sobolev space $W^{m,\alpha}$. In short hand, when $\alpha = 2$ we write (3.4) as

$$\|\pi w - w\| \leq C_1 h^m \|w\|_m, \quad \text{and} \quad \|\pi \mathbf{z} - \mathbf{z}\| \leq C_2 h^m \|\mathbf{z}\|_m.$$

(iii) There exists a positive C_3 such that

$$\|\Pi \mathbf{v} - \mathbf{v}\|_{L^\alpha(\Omega)} \leq C_3 h^m \|\mathbf{v}\|_{m,\alpha} \quad (3.5)$$

for any $\mathbf{v} \in (W^{m,\alpha}(\Omega))^d$, $1/\alpha \leq m \leq r+1$, $1 \leq \alpha \leq \infty$.

Because of the commuting relation between π, Π and the divergence (i.e., that $\nabla \cdot \Pi \mathbf{u} = \pi(\nabla \cdot \mathbf{u})$), we also have the bound

$$\|\nabla \cdot (\Pi \mathbf{v} - \mathbf{v})\|_{L^\alpha(\Omega)} \leq C_1 h^m \|\nabla \cdot \mathbf{v}\|_{m,\alpha}, \quad (3.6)$$

provided $\nabla \cdot \mathbf{v} \in W^{m,\alpha}(\Omega)$ for $1 \leq m \leq r+1$.

The semidiscrete expanded mixed formulation of (3.3) can read as following: Find $(p_h, \mathbf{u}_h, \mathbf{s}_h) : [0, T] \rightarrow W_h \times V_h \times \tilde{W}_h$ such that

$$(\bar{p}_{h,t}, w_h) + (\nabla \cdot \mathbf{u}_h, w_h) = (f - \Psi_t, w_h), \quad \forall w_h \in W_h, \quad (3.7a)$$

$$(\mathbf{u}_h, \mathbf{z}_h) + (K(|\mathbf{s}_h|)\mathbf{s}_h, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (3.7b)$$

$$(\mathbf{s}_h, \mathbf{v}_h) + (\bar{p}_h, \nabla \cdot \mathbf{v}_h) = (\nabla \Psi, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \quad (3.7c)$$

where $\bar{p}_h(x, 0) = \pi \bar{p}_0(x)$ and $\bar{p}_h = p_h - \pi \Psi$.

Fully discrete method. We use backward Euler for time-difference discretization. Let N be the positive integer, $t_0 = 0 < t_1 < \dots < t_N = T$ be partition interval $[0, T]$ of N sub-intervals, and let $\Delta t = t_n - t_{n-1} = T/N$ be the n -th time step size, $t_n = n\Delta t$ and $\varphi^n = \varphi(\cdot, t_n)$.

The discrete time expanded mixed finite element approximation to (3.3) is defined as follows: Find $(p_h^n, \mathbf{u}_h^n, \mathbf{s}_h^n) \in W_h \times V_h \times \tilde{W}_h$, $n = 1, 2, \dots, N$, such that

$$\left(\frac{\bar{p}_h^n - \bar{p}_h^{n-1}}{\Delta t}, w_h \right) + (\nabla \cdot \mathbf{u}_h^n, w_h) = (f^n - \Psi_t^n, w_h), \quad \forall w_h \in W_h, \quad (3.8a)$$

$$(\mathbf{u}_h^n, \mathbf{z}_h) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (3.8b)$$

$$(\mathbf{s}_h^n, \mathbf{v}_h) + (\bar{p}_h^n, \nabla \cdot \mathbf{v}_h) = (\nabla \Psi^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h. \quad (3.8c)$$

The initial approximations are chosen as follows:

$$\bar{p}_h^0(x) = \pi \bar{p}_0(x), \quad \mathbf{s}_h^0(x) = \pi \nabla p^0(x), \quad \mathbf{u}_h^0(x) = K(|\mathbf{s}_h^0(x)|)\mathbf{s}_h^0(x)$$

for all $x \in \Omega$.

4. Estimates of solutions . Using the theory of monotone operators [24, 35, 37], the authors in [17] proved the global existence of weak solution $p(x, t)$ of equation (3.2). Furthermore this solution is unique and belongs to $C([0, T], L^\alpha(\Omega))$, $\alpha \geq 1$ and $L_{loc}^\beta([0, T], W^{1,\beta}(\Omega))$, $p_t \in L_{loc}^{\beta'}([0, T], (W^{1,\beta}(\Omega))') \cap L_{loc}^2([0, T], L^2(\Omega))$ provided the initial data $p_0(x) \in L^2(\Omega) \cap W^{1,\beta}(\Omega)$ and Ψ, f sufficiently smooth. In fact, in [19, ?] the authors show that $p(x; t) \in L^\infty((0, T); L^\infty(\Omega)) \cap L^\infty((0, T); W^{1,\beta}(\Omega))$ and $p_t(x; t) \in L_{loc}^\infty((0, T); L^2(\Omega))$. Our aim explores the properties of the solutions, we assume that $p(x; t)$, initial data and boundary data have sufficiently regularities both in x and t variables so that our calculations are valid.

We begin with the Poincaré-Sobolev inequality with a specific weight which is essential in our estimate later.

LEMMA 4.1 (cf. [16]). *Let Ω be an open bounded domain in \mathbb{R}^d and $\xi(x) \geq 0$ be defined on Ω . Then for any function $u(x)$ vanishing on the boundary $\partial\Omega$ there is a positive constant C depending of Ω , $\deg(g)$ and coefficients of g such that.*

$$\|u\|_{L^{\beta^*}}^2 \leq C \left\| K^{\frac{1}{2}}(\xi) \nabla u \right\|^2 \left(1 + \left\| K^{\frac{1}{2}}(\xi) \xi \right\|^2 \right)^\gamma, \quad (4.1)$$

where $\beta^* = \frac{d\beta}{d-\beta}$.

THEOREM 4.2. *Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3).*

(i) *There is a positive constant C such that for any $t \in (0, T)$,*

$$\|\bar{p}(t)\|^2 + \int_0^t \left\| K^{\frac{1}{2}}(|\mathbf{s}(\tau)|) \mathbf{s}(\tau) \right\|^2 d\tau \leq \|\bar{p}^0\|^2 + C \int_0^t A(\tau) d\tau, \quad (4.2)$$

where

$$A = A(t) = \|\nabla \Psi(t)\|^2 + \|(f - \Psi_t)(t)\|_{L^r(\Omega)} + \|(f - \Psi_t)(t)\|_{L^r(\Omega)}^{\frac{\beta}{\beta-1}}$$

with $r = \frac{d\beta}{\beta(d+1)-d}$. Consequently,

$$\|p(t)\|^2 + \int_0^t \left\| K^{\frac{1}{2}}(|\mathbf{s}(\tau)|) \mathbf{s}(\tau) \right\|^2 d\tau \leq \|\bar{p}^0\|^2 + \|\Psi\|^2 + C \int_0^t A(\tau) d\tau.$$

(ii) *There exist positive constants C, C_1 such that for any $t \in (0, T)$,*

$$\|\mathbf{u}(t)\|^2 + \|\mathbf{s}(t)\|_{L^\beta(\Omega)}^\beta \leq C \left(\|\bar{p}^0\|^2 + 1 \right) + C \int_0^t e^{-C_1(t-\tau)} (\Lambda + B)(\tau) d\tau, \quad (4.3)$$

where

$$\Lambda = \Lambda(t) = \int_0^t A(\tau) d\tau, \quad (4.4)$$

$$B = B(t) = A(t) + \|\nabla \Psi_t(t)\|^2 + \|(\Psi_t - f)(t)\|^2. \quad (4.5)$$

Proof. (i) By selecting $w = \bar{p}$, $\mathbf{z} = \mathbf{s}$ and $\mathbf{v} = \mathbf{u}$ at each time level in (3.3) we have

$$\begin{aligned} (\bar{p}_t, \bar{p}) + (\nabla \cdot \mathbf{u}, \bar{p}) &= (f - \Psi_t, \bar{p}), \\ (\mathbf{u}, \mathbf{s}) + (K(|\mathbf{s}|) \mathbf{s}, \mathbf{s}) &= 0, \\ (\mathbf{s}, \mathbf{u}) + (\bar{p}, \nabla \cdot \mathbf{u}) &= (\nabla \Psi, \mathbf{u}). \end{aligned}$$

Adding three above equations implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{p}\|^2 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 &= (f - \Psi_t, \bar{p}) - (\nabla \Psi, \mathbf{u}) \\ &\leq (f - \Psi_t, \bar{p}) + \frac{1}{2} (\|\nabla \Psi\|^2 + \|\mathbf{u}\|^2). \end{aligned} \quad (4.6)$$

Using (3.3b) with $\mathbf{z} = \mathbf{u} \in \tilde{W}$, we have

$$\|\mathbf{u}\|^2 = -(K(|\mathbf{s}|)\mathbf{s}, \mathbf{u}) \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| \|\mathbf{u}\|,$$

which yields

$$\|\mathbf{u}\| \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|. \quad (4.7)$$

Thus (4.6) and (4.7) give

$$\frac{d}{dt} \|\bar{p}\|^2 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 \leq 2(f - \Psi_t, \bar{p}) + \|\nabla \Psi\|^2. \quad (4.8)$$

By Höder's inequality and (4.1),

$$\begin{aligned} (f - \Psi_t, \bar{p}) &\leq \|f - \Psi_t\|_{L^r(\Omega)} \|\bar{p}\|_{L^{\beta^*}} \\ &\leq C \|f - \Psi_t\|_{L^r(\Omega)} \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\nabla \bar{p} \right\| \left(1 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| \right)^\gamma. \end{aligned} \quad (4.9)$$

To estimate the second term on the right hand side of (4.9) we integrate by part (3.3c) and then select $\mathbf{v} = K(|\mathbf{s}|)\nabla \bar{p} \in V$. It follows that

$$(\nabla \bar{p}, K(|\mathbf{s}|)\nabla \bar{p}) = (\mathbf{s}, K(|\mathbf{s}|)\nabla \bar{p}) - (\nabla \Psi, K(|\mathbf{s}|)\nabla \bar{p})$$

which shows that

$$\left\| K^{\frac{1}{2}}(|\mathbf{s}|)\nabla \bar{p} \right\|^2 = (\mathbf{s} - \nabla \Psi, K(|\mathbf{s}|)\nabla \bar{p}) \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)(\mathbf{s} - \nabla \Psi) \right\| \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\nabla \bar{p} \right\|.$$

This, triangle inequality and the upper boundedness of $K(\cdot)$ give

$$\left\| K^{\frac{1}{2}}(|\mathbf{s}|)\nabla \bar{p} \right\| \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| + C \|\nabla \Psi\|. \quad (4.10)$$

Combining (4.9), (4.10) and Young's inequality yield

$$\begin{aligned} (f - \Psi_t, \bar{p}) &\leq C \|f - \Psi_t\|_{L^r(\Omega)} \left(\left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| + \|\nabla \Psi\| \right) \left(1 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| \right)^\gamma \\ &\leq C \|f - \Psi_t\|_{L^r(\Omega)} \left(1 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^{\gamma+1} + \|\nabla \Psi\|^{\gamma+1} \right) \\ &\leq \frac{1}{2} \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 + CA(t). \end{aligned} \quad (4.11)$$

It follows from (4.8) and (4.11) that

$$\frac{d}{dt} \|\bar{p}\|^2 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 \leq CA(t). \quad (4.12)$$

Integrating (4.12) from 0 to t we obtain (4.2).

(ii) Choosing $w = \bar{p}_t$, $\mathbf{z} = \mathbf{s}_t$ in (3.3a), (3.3b), differentiating (3.3c) with respect t and selecting $\mathbf{v} = \mathbf{u}$ we find that

$$\begin{aligned} (\bar{p}_t, \bar{p}_t) + (\nabla \cdot \mathbf{u}, \bar{p}_t) &= (f - \Psi_t, \bar{p}_t), \\ (\mathbf{u}, \mathbf{s}_t) + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_t) &= 0, \\ (\mathbf{s}_t, \mathbf{u}) + (\bar{p}_t, \nabla \cdot \mathbf{u}) &= (\nabla \Psi_t, \mathbf{u}). \end{aligned}$$

Adding three resultant equations gives

$$\|\bar{p}_t\|^2 + (K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_t) = (f - \Psi_t, \bar{p}_t) - (\nabla \Psi_t, \mathbf{u}). \quad (4.13)$$

Using (3.3b) with $\mathbf{z} = \nabla \Psi_t \in \tilde{W}$ we have

$$(\mathbf{u}, \nabla \Psi_t) = -(K(|\mathbf{s}|)\mathbf{s}, \nabla \Psi_t) \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| \|\nabla \Psi_t\|.$$

Note that definition of function $H(\cdot)$ in (2.14) gives

$$K(|\mathbf{s}|)\mathbf{s} \cdot \mathbf{s}_t = \frac{1}{2} \frac{d}{dt} H(\mathbf{s}).$$

Thus, (4.13) yields

$$\begin{aligned} \|\bar{p}_t\|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} H(x, t) dx &\leq C \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\| \|\nabla \Psi_t\| + \|f - \Psi_t\| \|\bar{p}_t\| \\ &\leq \varepsilon \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 + C_{\varepsilon} \|\nabla \Psi_t\|^2 + \frac{1}{2} (\|f - \Psi_t\|^2 + \|\bar{p}_t\|^2) \end{aligned} \quad (4.14)$$

for all $\varepsilon > 0$, where $H(x, t) = H(\mathbf{s}(x, t))$.

Adding (4.12) and (4.14) and selecting sufficiently small ε implies

$$\|\bar{p}_t\|^2 + \frac{d}{dt} \int_{\Omega} H(x, t) dx + (\bar{p}, \bar{p}_t) + C \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 \leq CB(t).$$

Then by Cauchy's inequality:

$$\frac{1}{2} \|\bar{p}_t\|^2 + \frac{d}{dt} \int_{\Omega} H(x, t) dx \leq -C \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s} \right\|^2 + \frac{1}{2} \|\bar{p}\|^2 + CB(t).$$

This and (2.15) show that

$$\frac{1}{2} \|\bar{p}_t\|^2 + \frac{d}{dt} \int_{\Omega} H(x, t) dx \leq -C \int_{\Omega} H(x, t) dx + \frac{1}{2} \|\bar{p}\|^2 + CB(t).$$

Ignoring the first term and using Gronwall's inequality we obtain

$$\begin{aligned} \int_{\Omega} H(x, t) dx &\leq -e^{-C_1 t} \int_{\Omega} H(x, 0) dx \\ &\quad + C \int_0^t e^{-C_1(t-\tau)} (\|\bar{p}\|^2 + B) d\tau. \end{aligned} \quad (4.15)$$

Using (2.11), (2.15) and (4.2), we have from (4.15) that

$$\begin{aligned} \|\mathbf{s}\|_{L^{\beta}(\Omega)}^{\beta} &\leq -e^{-C_1 t} \|\mathbf{s}^0\|_{L^{\beta}(\Omega)}^{\beta} + C + C \int_0^t e^{-C_1(t-\tau)} (\|\bar{p}^0\|^2 + \Lambda + B) d\tau \\ &\leq C (\|\bar{p}^0\|^2 + 1) + C \int_0^t e^{-C_1(t-\tau)} (\Lambda + B) d\tau. \end{aligned} \quad (4.16)$$

In addition, due to (4.7) and $K(\xi)\xi^2 \leq C\xi^{2-a} = C\xi^\beta$,

$$\|\mathbf{u}\|^2 \leq \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta. \quad (4.17)$$

Combining (4.17) and (4.16), we obtain (4.3). The proof is complete. \square

Although solution is considered continuous at $t = 0$ in appropriate Lebesgue or Sobolev space. Its time derivative is not. In the following we prove the time derivative of pressure is bounded.

THEOREM 4.3. *There is a positive constant C such that for any $0 < t_0 \leq t \leq T$,*

$$\begin{aligned} \|\bar{p}_t(t)\|^2 \leq & C \left\{ t_0^{-1} \left(\|\bar{p}^0\|^2 + \int_0^{t_0} (\Lambda + B)(\tau) d\tau \right) \right. \\ & \left. + \int_0^t (\|f_t - \Psi_{tt}(\tau)\|^2 + \|\nabla \Psi_t(\tau)\|^2) d\tau \right\}, \end{aligned} \quad (4.18)$$

where $B(t)$ is given as in (4.5).

Proof. We differentiate (3.3) with respect t ,

$$(\bar{p}_{tt}, w) + (\nabla \cdot \mathbf{u}_t, w) = (f_t - \Psi_{tt}, w), \quad \forall w \in W, \quad (4.19a)$$

$$(\mathbf{u}_t, \mathbf{z}) + (K(|\mathbf{s}|)\mathbf{s}_t, \mathbf{z}) + \left(K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s}, \mathbf{z} \right) = 0, \quad \forall \mathbf{z} \in \tilde{W}, \quad (4.19b)$$

$$(\mathbf{s}_t, \mathbf{v}) + (\bar{p}_t, \nabla \cdot \mathbf{v}) = (\nabla \Psi_t, \mathbf{v}), \quad \forall \mathbf{v} \in V. \quad (4.19c)$$

Selecting $w = \bar{p}_t$, $\mathbf{z} = \mathbf{s}_t$ and $\mathbf{v} = \mathbf{u}_t$ and summing resultant equations we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{p}_t\|^2 + \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|^2 = & - \left(K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s}, \mathbf{s}_t \right) + (f_t - \Psi_{tt}, \bar{p}_t) \\ & - (\nabla \Psi_t, \mathbf{u}_t) = I_1 + I_2 + I_3. \end{aligned} \quad (4.20)$$

According to (2.12),

$$\left| K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s} \right| \leq aK(|\mathbf{s}|)|\mathbf{s}_t|,$$

which leads to

$$|I_1| \leq a \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|^2. \quad (4.21)$$

By Cauchy's inequality,

$$|I_2| \leq \frac{1}{2} \left(\|f_t - \Psi_{tt}\|^2 + \|\bar{p}_t\|^2 \right). \quad (4.22)$$

For all $\varepsilon > 0$,

$$|I_3| \leq C\varepsilon^{-1} \|\nabla \Psi_t\|^2 + \varepsilon \|\mathbf{u}_t\|^2.$$

In (4.19b), taking $\mathbf{z} = \mathbf{u}_t$ we find that

$$\|\mathbf{u}_t\| \leq \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\| + \left\| K'(|\mathbf{s}|) \frac{\mathbf{s} \cdot \mathbf{s}_t}{|\mathbf{s}|} \mathbf{s} \right\| \leq (1 + a) \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|. \quad (4.23)$$

By choosing $\varepsilon = \frac{1-a}{2(1+a)} > 0$,

$$|I_3| \leq C \|\nabla \Psi_t\|^2 + \frac{1-a}{2} \left\| K^{\frac{1}{2}}(|\mathbf{s}_h|)\mathbf{s}_t \right\|^2. \quad (4.24)$$

It follows from (4.20), (4.21), (4.22) and (4.24) that

$$\frac{d}{dt} \|p_t\|^2 + (1-a) \left\| K^{\frac{1}{2}}(|\mathbf{s}|)\mathbf{s}_t \right\|^2 \leq \|f_t - \Psi_{tt}\|^2 + \|\bar{p}_t\|^2 + C \|\nabla \Psi_t\|^2.$$

Dropping the nonnegative term of the left hand side gives

$$\frac{d}{dt} \|\bar{p}_t\|^2 \leq \|\bar{p}_t\|^2 + C(\|f_t - \Psi_{tt}\|^2 + \|\nabla \Psi_t\|^2). \quad (4.25)$$

For $t \geq t' > 0$, applying Gronwall's inequality to (4.25) we find that

$$\|\bar{p}_t\|^2 \leq C \left(\|\bar{p}_{t'}\|^2 + \int_{t'}^t (\|f_t - \Psi_{tt}\|^2 + \|\nabla \Psi_t\|^2) d\tau \right). \quad (4.26)$$

Integrating (4.26) in t' from 0 to t_0 , using (4.15) we obtain

$$\begin{aligned} t_0 \|\bar{p}_t\|^2 &\leq C \left(\int_0^{t_0} \|\bar{p}_{t'}\|^2 + \int_0^{t_0} \int_{t'}^t (\|\Psi_{tt} - f_t\|^2 + \|\nabla \Psi_t\|^2) d\tau \right) \\ &\leq C \left(\int_0^{t_0} e^{-C_1(t_0-\tau)} (\|\bar{p}\|^2 + B) d\tau + t_0 \int_0^t (\|f_t - \Psi_{tt}\|^2 + \|\nabla \Psi_t\|^2) d\tau \right). \end{aligned}$$

Now using (4.2) we deduce previous inequality to

$$t_0 \|\bar{p}_t\|^2 \leq C \left(\|\bar{p}^0\|^2 + \int_0^{t_0} (\Lambda + B) d\tau + t_0 \int_0^t (\|f_t - \Psi_{tt}\|^2 + \|\nabla \Psi_t\|^2) d\tau \right)$$

which proves (4.18). The proof is complete. \square

We also obtain the similar results for solution of (3.7) as following.

THEOREM 4.4. *Let $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete problem (3.7).*

(i) *There is a positive constant C such that for each $t \in (0, T)$,*

$$\|\bar{p}_h(t)\|^2 + \int_0^t \left\| K^{\frac{1}{2}}(|\mathbf{s}_h(\tau)|)\mathbf{s}_h(\tau) \right\|^2 d\tau \leq \|\bar{p}^0\|^2 + C \int_0^t A(\tau) d\tau. \quad (4.27)$$

(ii) *There are two positive constants C, C_1 such that for each $t \in (0, T)$*

$$\|\mathbf{u}_h(t)\|^2 + \|\mathbf{s}_h(t)\|_{L^\beta(\Omega)}^\beta \leq C \left(\|\bar{p}^0\|^2 + 1 + \int_0^t e^{-C_1(t-\tau)} (\Lambda + B)(\tau) d\tau \right). \quad (4.28)$$

(iii) *For $0 < t_0 \leq t \leq T$,*

$$\begin{aligned} \|\bar{p}_{h,t}\|^2 &\leq C \left\{ t_0^{-1} \left(\|\bar{p}^0\|^2 + \int_0^{t_0} (\Lambda + B)(\tau) d\tau \right) \right. \\ &\quad \left. + \int_0^t (\|f_t - \Psi_{tt}(\tau)\|^2 + \|\nabla \Psi_t(\tau)\|^2) d\tau \right\}, \end{aligned} \quad (4.29)$$

where C is a positive constant.

To finish this section we give a bound of pressure in L^∞ -norm which is useful for our error estimate in the later sections.

THEOREM 4.5. *Let $(p, \mathbf{u}, \mathbf{s})$ solve the problem (3.3), $\mu \geq 2$ and $\mu > \frac{ad}{\beta}$. If $T > 0$ then*

$$\begin{aligned} \sup_{t \in [0, T]} \|\bar{p}(t)\|_{L^\infty(\Omega)} &\leq 2\|\bar{p}^0\|_{L^\infty(\Omega)} \\ &+ C \left\{ (1+T)^\mu \left(1 + \sup_{t \in [0, T]} \|(f - \Psi_t)(t)\|_{L^{\mu+1}(\Omega)}^\mu + \sup_{t \in [0, T]} \|\nabla \Psi(t)\|_{L^\infty(\Omega)}^{\frac{\beta\mu}{2}} \right) \right\}^{\frac{1}{\mu-a}}. \end{aligned} \quad (4.30)$$

The proof of Theorem 4.5 is given in Appendix.

5. Error analysis. In this section, we use estimates in the previous section, the techniques in [15] and the expanded mixed finite element method to establish the error estimates between the analytical solution and approximation solution in several norms.

5.1. Error estimate for semidiscrete method. We will bound the error in the semidiscrete method in various norms by comparing the computed solution to the projections of the true solutions. To do this, we restrict the test functions in (3.3) to the finite-dimensional spaces. Let

$$\begin{aligned} \bar{p}_h - \bar{p} &= (\bar{p}_h - \pi\bar{p}) + (\pi\bar{p} - \bar{p}) \equiv \vartheta + \theta, \\ \mathbf{s}_h - \mathbf{s} &= (\mathbf{s}_h - \pi\mathbf{s}) + (\pi\mathbf{s} - \mathbf{s}) \equiv \eta + \zeta, \\ \mathbf{u}_h - \mathbf{u} &= (\mathbf{u}_h - \Pi\mathbf{u}) + (\Pi\mathbf{u} - \mathbf{u}) \equiv \rho + \varrho. \end{aligned}$$

Properties of projections in (3.4) and (3.5) yield for each $t \in [0, T]$,

$$\|\theta\|_{L^\alpha} \leq Ch^m \|\bar{p}\|_{m, \alpha}, \quad \forall \bar{p} \in W^{m, \alpha}(\Omega), \quad (5.1)$$

$$\|\zeta\|_{L^\alpha} \leq Ch^m \|\mathbf{s}\|_{m, \alpha}, \quad \forall \mathbf{s} \in (W^{m, \alpha}(\Omega))^d, \quad (5.2)$$

$$\|\varrho\|_{L^\alpha} \leq Ch^m \|\mathbf{u}\|_{m, \alpha}, \quad \forall \mathbf{u} \in (W^{m, \alpha}(\Omega))^d. \quad (5.3)$$

for all $1 \leq m \leq r+1$, $1 \leq \alpha \leq \infty$. Let $t_0 > 0$,

$$\begin{aligned} \Upsilon &= 1 + \|\bar{p}^0\|^2 + \sup_{t \in [0, T]} \int_0^t e^{-C_1(t-\tau)} (\Lambda + B)(\tau) d\tau, \\ \Xi &= t_0^{-1} \left(\|\bar{p}^0\|^2 + \int_0^{t_0} (\Lambda + B)(\tau) d\tau \right) + \int_0^T (\|(f_t - \Psi_{tt})(\tau)\|^2 + \|\nabla \Psi_t(\tau)\|^2) d\tau. \end{aligned}$$

where $\Lambda(t)$ and $B(t)$ are defined in (4.4), (4.5).

THEOREM 5.1. *Assume $(\bar{p}^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(\bar{p}_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete mixed finite element approximation (3.7). There is a positive constant C such that for any $t \in (0, T)$,*

$$\|(p_h - p)(t)\|^2 \leq C \left(\|\theta(t)\|^2 + \|(\pi\Psi - \Psi)(t)\|^2 \right) + C\Upsilon \int_0^t \|\zeta(\tau)\|_{L^\beta(\Omega)} d\tau. \quad (5.4)$$

Consequently, if $p, \Psi \in L^\infty(0, T; H^{r+1}(\Omega))$, $\mathbf{s} \in L^2(0, T; (W^{r+1, \beta}(\Omega))^d)$ then for any $t \in (0, T)$,

$$\begin{aligned} \|(p_h - p)(t)\| &\leq Ch^{r+1} (\|\bar{p}(t)\|_{r+1} + \|\Psi(t)\|_{r+1}) \\ &\quad + C\Upsilon^{\frac{1}{2}} h^{\frac{r+1}{2}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1, \beta}^2 d\tau}. \end{aligned} \quad (5.5)$$

Proof. Subtracting (3.7) from (3.3) we obtain the equations of difference:

$$(\bar{p}_{h,t} - \bar{p}_t, w_h) + (\nabla \cdot (\mathbf{u}_h - \mathbf{u}), w_h) = 0, \quad \forall w_h \in W_h, \quad (5.6a)$$

$$(\mathbf{u}_h - \mathbf{u}, \mathbf{z}_h) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (5.6b)$$

$$(\mathbf{s}_h - \mathbf{s}, \mathbf{v}_h) + (\bar{p}_h - \bar{p}, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h. \quad (5.6c)$$

Let $w_h = \vartheta$, $\mathbf{z}_h = \eta$ and $\mathbf{v}_h = \rho$. Using the L^2 -projection and $H(\text{div})$ -projection, we have from (5.6a)–(5.6c) that

$$(\vartheta_t, \vartheta) + (\nabla \cdot \rho, \vartheta) = 0, \quad (5.7a)$$

$$(\rho, \eta) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \eta) = 0, \quad (5.7b)$$

$$(\eta, \rho) + (\vartheta, \nabla \cdot \rho) = 0. \quad (5.7c)$$

Adding three equations (5.7a)–(5.7c) gives

$$\frac{1}{2} \frac{d}{dt} \|\vartheta\|^2 + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \eta) = 0$$

or

$$\frac{1}{2} \frac{d}{dt} \|\vartheta\|^2 + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_h - \mathbf{s}) = (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \zeta). \quad (5.8)$$

Applying (2.18) to the second term of (5.8) we have

$$(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_h - \mathbf{s}) \geq C\omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2, \quad (5.9)$$

where

$$\omega = \omega(t) = \left(1 + \max\{\|\mathbf{s}_h(t)\|_{L^\beta(\Omega)}, \|\mathbf{s}(t)\|_{L^\beta(\Omega)}\}\right)^{-a}. \quad (5.10)$$

Since $K(|\xi|)\xi \leq C\xi^{\beta-1}$, the last term of (5.8) is bounded by

$$\begin{aligned} |(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \zeta)| &\leq C (|\mathbf{s}_h|^{\beta-1} + |\mathbf{s}|^{\beta-1}, |\zeta|) \\ &\leq C \left(\|\mathbf{s}_h\|_{L^\beta(\Omega)}^{\beta-1} + \|\mathbf{s}\|_{L^\beta(\Omega)}^{\beta-1} \right) \|\zeta\|_{L^\beta(\Omega)}. \end{aligned} \quad (5.11)$$

The last inequality is obtained by applying Höder's inequality with powers $\frac{\beta}{\beta-1}$ and β .

It follows from (5.8), (5.9) and (5.11) that

$$\begin{aligned} \frac{d}{dt} \|\vartheta\|^2 + \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq \left(\|\mathbf{s}_h\|_{L^\beta(\Omega)}^{\beta-1} + \|\mathbf{s}\|_{L^\beta(\Omega)}^{\beta-1} \right) \|\zeta\|_{L^\beta(\Omega)} \\ &\leq C \left(1 + \|\mathbf{s}_h\|_{L^\beta(\Omega)}^\beta + \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta \right) \|\zeta\|_{L^\beta(\Omega)}. \end{aligned} \quad (5.12)$$

Due to (4.3) and (4.28),

$$1 + \|\mathbf{s}_h\|_{L^\beta(\Omega)}^\beta + \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta \leq C\Upsilon. \quad (5.13)$$

Integrating (5.12) in time, using $\vartheta(0) = 0$, we have

$$\|\vartheta\|^2 + \int_0^t \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 d\tau \leq C\Upsilon \int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau. \quad (5.14)$$

Since

$$p_h - p = (\bar{p}_h - \bar{p}) + (\pi\Psi - \Psi) = \vartheta + \theta + (\pi\Psi - \Psi), \quad (5.15)$$

the inequality (5.4) follows from (5.15), Minkowski's inequality and (5.14).

Applying (5.2), (5.3) to (5.4) we obtain (5.5). The proof is complete. \square

The L^2 -error estimate and the inverse estimates enable us to find the L^∞ -error estimate as following:

THEOREM 5.2. *Assume $(\bar{p}^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(\bar{p}_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete mixed finite element approximation (3.7). If $p, \Psi \in L^\infty(0, T; W^{r+1, \infty}(\Omega))$, then there is a positive constant C such that for any $t \in (0, T)$,*

$$\begin{aligned} \|(p - p_h)(t)\|_{L^\infty(\Omega)} &\leq \|\theta(t)\|_{L^\infty(\Omega)} + \|(\pi\Psi - \Psi)(t)\|_{L^\infty(\Omega)} \\ &\quad + C\Upsilon^{\frac{1}{2}} h^{-1} \sqrt{\int_0^t \|\zeta(\tau)\|_{L^\beta(\Omega)} d\tau}. \end{aligned} \quad (5.16)$$

Furthermore if $\mathbf{s} \in L^1(0, T; (W^{r+1, \beta}(\Omega))^d)$ then

$$\begin{aligned} \|(p - p_h)(t)\|_{L^\infty(\Omega)} &\leq Ch^{r+1} \left(\|\bar{p}(t)\|_{r+1, \infty} + \|\Psi(t)\|_{r+1, \infty} \right) \\ &\quad + C\Upsilon^{\frac{1}{2}} h^{\frac{r-1}{2}} \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1, \beta} d\tau}. \end{aligned} \quad (5.17)$$

Proof. We have from (5.15) and triangle inequality that

$$\|p - p_h\|_{L^\infty} \leq \|\theta\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\pi\Psi - \Psi\|_{L^\infty}. \quad (5.18)$$

Due to the quasi-uniformly of \mathcal{T}_h , the following inverse estimate holds

$$\|\vartheta\|_{L^\infty} \leq Ch^{-\frac{2}{q}} \|\vartheta\|_{L^q} \quad \text{for all } 1 \leq q \leq \infty.$$

Applying this with $q = 2$ and using (5.14) imply

$$\begin{aligned} \|\vartheta\|_{L^\infty(\Omega)} &\leq Ch^{-1} \|\vartheta\| \leq C\Upsilon^{\frac{1}{2}} h^{-1} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}} \\ &\leq C\Upsilon^{\frac{1}{2}} h^{-1} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}}. \end{aligned} \quad (5.19)$$

Hence (5.16) follows from (5.18) and (5.19).

Using (5.16), (5.1) and (5.2) we obtain (5.17). \square Now we give the bound of $\|p - p_h\|_{H^{-1}}$ defined by

$$\|\cdot\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega)} \frac{(\cdot, \varphi)}{\|\varphi\|_{H^1(\Omega)}}.$$

LEMMA 5.3. *Under the assumption of Theorem 5.1 we have*

$$\begin{aligned} \left\| \int_0^t \nabla \cdot (\mathbf{u}_h - \mathbf{u})(\tau) d\tau \right\| &\leq C \Upsilon^{\frac{1}{2}} \sqrt{\int_0^t \|\zeta(\tau)\|_{L^\beta(\Omega)} d\tau} \\ &+ \left\| \int_0^t \nabla \cdot (\mathbf{u} - \Pi \mathbf{u})(\tau) d\tau \right\|. \end{aligned} \quad (5.20)$$

Proof. The L^2 -projection allows us to write $(\bar{p}_h - \bar{p}, w_h) = (\vartheta, w_h)$. Integrating (5.6a) from 0 to t we have

$$(\vartheta, w_h) + \left(\int_0^t \nabla \cdot \rho d\tau, w_h \right) = (\vartheta(0), w_h) = 0 \quad (5.21)$$

for all $w_h \in W_h$. Now choose $w_h = \int_0^t \nabla \cdot \rho d\tau \in W_h$ then

$$\left\| \int_0^t \nabla \cdot \rho d\tau \right\| \leq \|\vartheta\|.$$

Triangle inequality yields

$$\left\| \int_0^t \nabla \cdot (\mathbf{u}_h - \mathbf{u}) d\tau \right\| \leq \|\vartheta\| + \left\| \int_0^t \nabla \cdot \varrho d\tau \right\|.$$

Using (5.14) we obtain (5.20). \square

THEOREM 5.4. *Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete mixed finite element approximation (3.7). There is a positive constant C such that for each $t \in (0, T)$,*

$$\begin{aligned} \|(\bar{p} - \bar{p}_h)(t)\|_{H^{-1}(\Omega)} &\leq C \left\{ h \|\theta(t)\| + h \|(\pi \Psi - \Psi)(t)\| \right. \\ &\left. + C \Upsilon^{\frac{1}{2}} \sqrt{\int_0^t \|\zeta(\tau)\|_{L^\beta(\Omega)} d\tau} + \int_0^t \|\nabla \cdot (\mathbf{u} - \Pi \mathbf{u})(\tau)\| d\tau \right\}. \end{aligned} \quad (5.22)$$

Proof. Let $\varphi \in H_0^1(\Omega)$ and $\pi \varphi \in W_h$

$$(\bar{p} - \bar{p}_h, \varphi) = (\bar{p} - \bar{p}_h, \varphi - \pi \varphi) + (\bar{p} - \bar{p}_h, \pi \varphi). \quad (5.23)$$

Properties of projection allow us to bound

$$(\bar{p} - \bar{p}_h, \varphi - \pi \varphi) \leq C \|\bar{p} - \bar{p}_h\| \|\varphi - \pi \varphi\| \leq Ch \|\bar{p} - \bar{p}_h\| \|\varphi\|_{H^1}. \quad (5.24)$$

Since $\int_0^t \nabla \cdot \rho d\tau \in W_h$, the L^2 -project shows that

$$\left(\int_0^t \nabla \cdot \rho d\tau, \pi \varphi \right) = \left(\int_0^t \nabla \cdot \rho d\tau, \varphi \right).$$

Using (5.6a) with $w_h = \pi\varphi$ and definition of projections we find that

$$\begin{aligned} (\vartheta, \pi\varphi) &= \int_0^t (\vartheta_t, \pi\varphi) d\tau = \int_0^t (\bar{p}_{h,t} - \bar{p}_t, \pi\varphi) d\tau \\ &= - \left(\int_0^t \nabla \cdot (\mathbf{u}_h - \mathbf{u}) d\tau, \pi\varphi \right) = - \left(\int_0^t \nabla \cdot \rho d\tau, \pi\varphi \right) \\ &= - \left(\int_0^t \nabla \cdot \rho d\tau, \varphi \right). \end{aligned}$$

Thus

$$(\bar{p} - \bar{p}_h, \pi\varphi) = -(\vartheta, \pi\varphi) \leq C \left\| \int_0^t \nabla \cdot \rho d\tau \right\| \|\varphi\|_{H^1}. \quad (5.25)$$

It follows from (5.23), (5.24) and (5.25) that

$$\begin{aligned} \frac{(\bar{p} - \bar{p}_h, \varphi)}{\|\varphi\|_{H^1}} &\leq Ch \|\bar{p} - \bar{p}_h\| + \left\| \int_0^t \nabla \cdot \rho d\tau \right\| \\ &\leq Ch \|\bar{p} - \bar{p}_h\| + \left\| \int_0^t \nabla \cdot (\mathbf{u}_h - \mathbf{u}) d\tau \right\| + \left\| \int_0^t \nabla \cdot (\Pi\mathbf{u} - \mathbf{u}) d\tau \right\|. \end{aligned}$$

This, (5.20) and (5.4) implies (5.22). \square

Return to error estimate for vector gradient of pressure we have the following results

THEOREM 5.5. *Under the assumptions of Theorem 5.1. There exists a positive constant C independent of h such that for each $0 < t_0 \leq t \leq T$,*

$$\|(\mathbf{s}_h - \mathbf{s})(t)\|_{L^\beta(\Omega)}^2 \leq C\Upsilon^{\gamma+\frac{1}{2}} \Xi \sqrt{\int_0^t \|\zeta(\tau)\|_{L^\beta(\Omega)} d\tau} + \Upsilon^{\gamma+1} \|\zeta(t)\|_{L^\beta(\Omega)}. \quad (5.26)$$

Consequently, if $\mathbf{s} \in L^1(0, T; (W^{r+1, \beta}(\Omega))^d)$ then

$$\begin{aligned} \|(\mathbf{s}_h - \mathbf{s})(t)\|_{L^\beta(\Omega)} &\leq C\Upsilon^{\frac{2\gamma+1}{4}} \Xi^{\frac{1}{2}} h^{\frac{r+1}{4}} \left(\int_0^t \|\mathbf{s}(\tau)\|_{r+1, \beta} d\tau \right)^{\frac{1}{4}} \\ &\quad + C\Upsilon^{\frac{\gamma+1}{2}} h^{\frac{r+1}{2}} \sqrt{\|\mathbf{s}(t)\|_{r+1, \beta}} \end{aligned} \quad (5.27)$$

and

$$\begin{aligned} \|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^\beta(\Omega)} &\leq C\Upsilon^{\frac{2\gamma+1}{4}} \Xi^{\frac{1}{2}} h^{\frac{r+1}{4}} \left(\int_0^t \|\mathbf{s}(\tau)\|_{r+1, \beta} d\tau \right)^{\frac{1}{4}} \\ &\quad + C\Upsilon^{\frac{\gamma+1}{2}} h^{\frac{r+1}{2}} \sqrt{\|\mathbf{s}(t)\|_{r+1, \beta}} + Ch^{r+1} \|\mathbf{u}(t)\|_{r+1, \beta}. \end{aligned} \quad (5.28)$$

Proof. Thank to (5.9), (5.8) and L^2 -projection,

$$\begin{aligned} \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \mathbf{s}_h - \mathbf{s}) \\ &= -(\bar{p}_{h,t} - \bar{p}_t, \vartheta) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \zeta), \end{aligned}$$

from which, (5.11), (5.13) and (5.14). It follows that

$$\begin{aligned} \omega \|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 &\leq C(\|\bar{p}_{h,t}\| + \|\bar{p}_t\|) \|\vartheta\| + C\Upsilon \|\zeta\|_{L^\beta(\Omega)} \\ &\leq C\Xi \left(\Upsilon \int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}} + C\Upsilon \|\zeta\|_{L^\beta(\Omega)}. \end{aligned} \quad (5.29)$$

Thus

$$\|\mathbf{s}_h - \mathbf{s}\|_{L^\beta(\Omega)}^2 \leq C\Upsilon^{\frac{1}{2}}\Xi\omega^{-1} \left(\int_0^t \|\zeta\|_{L^\beta(\Omega)} d\tau \right)^{\frac{1}{2}} + C\Upsilon\omega^{-1} \|\zeta\|_{L^\beta(\Omega)}. \quad (5.30)$$

Note that from (5.10) we have

$$\begin{aligned} \omega^{-1} &\leq C \left(1 + \|\mathbf{s}\|_{L^\beta(\Omega)} + \|\mathbf{s}_h\|_{L^\beta(\Omega)} \right)^a \\ &\leq C \left(1 + \|\mathbf{s}\|_{L^\beta(\Omega)}^\beta + \|\mathbf{s}_h\|_{L^\beta(\Omega)}^\beta \right)^\gamma \leq C\Upsilon^\gamma. \end{aligned} \quad (5.31)$$

Substituting (5.31) into (5.30), we obtain (5.26).

Using (5.2) in (5.26) we obtain (5.27).

To prove (5.28) we use (5.6b) with $\mathbf{z}_h = \rho^{\beta-1} \in \tilde{W}_h$:

$$\|\rho\|_{L^\beta}^\beta = (\rho, \rho^{\beta-1}) = (\mathbf{u}_h - \mathbf{u}, \rho^{\beta-1}) = - (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|)\mathbf{s}, \rho^{\beta-1}).$$

Proposition 2.2 and Höder's inequality lead to

$$\|\rho\|_{L^\beta}^\beta \leq C(|\mathbf{s}_h - \mathbf{s}|, \rho^{\beta-1}) \leq C\|\mathbf{s} - \mathbf{s}_h\|_{L^\beta(\Omega)} \|\rho\|_{L^\beta(\Omega)}^{\beta-1}$$

and hence

$$\|\rho\|_{L^\beta(\Omega)} \leq C\|\mathbf{s} - \mathbf{s}_h\|_{L^\beta(\Omega)}. \quad (5.32)$$

Triangle inequality and (5.32) yield

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^\beta(\Omega)} \leq C(\|\rho\|_{L^\beta(\Omega)} + \|\varrho\|_{L^\beta(\Omega)}) \leq C(\|\mathbf{s} - \mathbf{s}_h\|_{L^\beta(\Omega)} + \|\varrho\|_{L^\beta(\Omega)}).$$

Therefore (5.28) follows by using (5.27) and (5.3). The proof is complete. \square

Non-degenerate case. In previous discussion, we developed error bounds based on the minimal regularity assumptions, using fairly weak norms on the error ($L^\beta(\Omega)$ -norm). In following discussion we bounds errors in numerical solution in term of strong norms (L^2 -norm), but make some assumption on the the regularity of solution. In particular, we assume that

$$p, \Psi \in L^\infty(0, T; H^{r+1}(\Omega)) \text{ and } \mathbf{s} \in L^\infty(0, T; (L^\infty(\Omega) \cap H^{r+1}(\Omega))^d).$$

THEOREM 5.6. *Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete mixed finite element approximation (3.7). Then there is a positive constant C such that for each $t \in (0, T)$,*

$$\begin{aligned} &\|(p_h - p)(t)\| + \sqrt{\int_0^t \|(\mathbf{s}_h - \mathbf{s})(\tau)\|^2 d\tau} \\ &\leq Ch^{r+1} \left\{ \|\Psi(t)\|_{r+1} + \|\bar{p}(t)\|_{r+1} + \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1}^2 d\tau} \right\}. \end{aligned} \quad (5.33)$$

and

$$\begin{aligned} \sqrt{\int_0^t \|(\mathbf{u}_h - \mathbf{u})(\tau)\|^2 d\tau} &\leq Ch^{r+1} \left\{ \|\Psi(t)\|_{r+1} + \|\bar{p}(t)\|_{r+1} \right. \\ &\quad \left. + \sqrt{\int_0^t \|\mathbf{u}(\tau)\|_{r+1}^2 + \|\mathbf{s}(\tau)\|_{r+1}^2 d\tau} \right\}. \end{aligned} \quad (5.34)$$

Proof. We use the equation (5.8). The regularity of solution enable us to bound term by term of equation (5.8) as following

According to (2.18),

$$(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \mathbf{s}_h - \mathbf{s}) \geq (1-a) \left\| K^{\frac{1}{2}}(\max\{|\mathbf{s}|, |\mathbf{s}_h|\})(\mathbf{s}_h - \mathbf{s}) \right\|^2.$$

Using the fact that $K(\cdot)$ is bounded from below, we find that

$$(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \mathbf{s}_h - \mathbf{s}) \geq k(1-a) \|\mathbf{s}_h - \mathbf{s}\|^2 \quad (5.35)$$

for some $k > 0$.

By Höder's inequality, (2.20) and Young's inequality,

$$\begin{aligned} (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta) &\leq C \|\mathbf{s}_h - \mathbf{s}\| \|\zeta\| \\ &\leq \frac{k(1-a)}{2} \|\mathbf{s}_h - \mathbf{s}\|^2 + C \|\zeta\|^2. \end{aligned} \quad (5.36)$$

Hence (5.35), (5.36) and (5.8) show that

$$\frac{d}{dt} \|\vartheta\|^2 + \|\mathbf{s}_h - \mathbf{s}\|^2 \leq C \|\zeta\|^2.$$

Integrating this from 0 to t , using $\vartheta(0) = 0$, we obtain

$$\|\vartheta\|^2 + \int_0^t \|\mathbf{s}_h - \mathbf{s}\|^2 d\tau \leq C \int_0^t \|\zeta\|^2 d\tau. \quad (5.37)$$

Thus

$$\|\bar{p}_h - \bar{p}\|^2 + \int_0^t \|\mathbf{s}_h - \mathbf{s}\|^2 d\tau \leq \|\theta\|^2 + C \int_0^t \|\zeta\|^2 d\tau. \quad (5.38)$$

Inequality (5.33) follows from (5.15), (5.1), (5.2) and (5.38).

In (5.7b), select $\mathbf{z}_h = \rho$ we obtain

$$\begin{aligned} \|\rho\|^2 &= -(K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \rho) \\ &\leq C(|\mathbf{s}_h - \mathbf{s}|, |\rho|) \\ &\leq C \|\mathbf{s}_h - \mathbf{s}\| \|\rho\|. \end{aligned} \quad (5.39)$$

This leads to

$$\begin{aligned} \int_0^t \|\mathbf{u}_h - \mathbf{u}\|^2 d\tau &\leq C \int_0^t \|\varrho\|^2 + \|\rho\|^2 d\tau \\ &\leq C \int_0^t \|\varrho\|^2 + \|\mathbf{s}_h - \mathbf{s}\|^2 d\tau. \end{aligned} \quad (5.40)$$

We obtain (5.34) by using (5.33) and (5.3) in (5.40). The proof is complete. \square

THEOREM 5.7. *Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h, \mathbf{u}_h, \mathbf{s}_h)$ solve the semidiscrete problem (3.7). If $p, \Psi \in L^\infty(0, T; W^{r+1, \infty}(\Omega))$ and $\mathbf{s} \in L^2(0, T; H^{r+1}(\Omega))^d$. Then there is a positive constant C such that for each $t \in (0, T)$,*

$$\begin{aligned} \|(p - p_h)(t)\|_{L^\infty(\Omega)} &\leq Ch^{r+1} \left(\|\bar{p}(t)\|_{r+1, \infty} + \|\Psi(t)\|_{r+1, \infty} \right) \\ &\quad + Ch^r \sqrt{\int_0^t \|\mathbf{s}(\tau)\|_{r+1}^2 d\tau}. \end{aligned}$$

Proof. It follows from (5.15), (5.37), (5.1) and (5.2) that

$$\begin{aligned} \|p - p_h\|_{L^\infty(\Omega)} &\leq \|\theta\|_{L^\infty(\Omega)} + \|\vartheta\|_{L^\infty(\Omega)} + \|\pi\Psi - \Psi\|_{L^\infty(\Omega)} \\ &\leq Ch^{r+1} \left(\|\bar{p}\|_{r+1, \infty} + \|\Psi\|_{r+1, \infty} \right) + Ch^{-1} \left(\int_0^t \|\zeta\|^2 d\tau \right)^{\frac{1}{2}} \\ &\leq Ch^{r+1} \left(\|\bar{p}\|_{r+1, \infty} + \|\Psi\|_{r+1, \infty} \right) + Ch^r \left(\int_0^t \|\mathbf{s}\|^2 d\tau \right)^{\frac{1}{2}} \end{aligned}$$

which completes the proof. \square

THEOREM 5.8. *Assume $(\bar{p}^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(\bar{p}_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Then there are positive constants C independent of h such that for each $0 < t_0 \leq t \leq T$,*

$$\|(\mathbf{s}_h - \mathbf{s})(t)\| \leq C\Xi^{\frac{1}{2}} h^{\frac{r+1}{2}} \left(\int_0^t \|\mathbf{s}(\tau)\|_{r+1}^2 d\tau \right)^{\frac{1}{4}} + Ch^{r+1} \|\mathbf{s}(t)\|_{r+1} \quad (5.41)$$

and

$$\begin{aligned} \|(\mathbf{u}_h - \mathbf{u})(t)\| &\leq C\Xi^{\frac{1}{2}} h^{\frac{r+1}{2}} \left(\int_0^t \|\mathbf{s}(\tau)\|_{r+1}^2 d\tau \right)^{\frac{1}{4}} \\ &\quad + Ch^{r+1} (\|\mathbf{s}(t)\|_{r+1} + \|\mathbf{u}(t)\|_{r+1}). \end{aligned} \quad (5.42)$$

Proof. Thank to (5.35), (5.8) and L^2 -projection,

$$\begin{aligned} C \|\mathbf{s}_h - \mathbf{s}\|^2 &\leq (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \mathbf{s}_h - \mathbf{s}) \\ &= -(\vartheta_t, \vartheta) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta) \\ &= -(\bar{p}_{h,t} - \bar{p}_t, \vartheta) + (K(|\mathbf{s}_h|)\mathbf{s}_h - K(|\mathbf{s}|\mathbf{s}), \zeta) \\ &\leq C(\|\bar{p}_{h,t}\| + \|\bar{p}_t\|) \|\vartheta\| + C \|\zeta\|^2. \end{aligned}$$

According to (4.18) and (4.29), $\|\bar{p}_{h,t}\| + \|\bar{p}_t\| \leq C\Xi$. This and (5.37) give

$$\|\mathbf{s}_h - \mathbf{s}\|^2 \leq C\Xi \left(\int_0^t \|\zeta\|^2 d\tau \right)^{\frac{1}{2}} + C \|\zeta\|^2. \quad (5.43)$$

Hence (5.41) holds.

We have

$$\|\mathbf{u}_h - \mathbf{u}\| \leq \|\rho\| + \|\varrho\|.$$

Using (5.39) and (5.43),

$$\|\rho\| \leq C \|\mathbf{s}_h - \mathbf{s}\| \leq C \Xi^{\frac{1}{2}} \left(\int_0^t \|\zeta\|^2 d\tau \right)^{\frac{1}{4}} + C \|\zeta\|.$$

This leads to

$$\|\mathbf{u}_h - \mathbf{u}\| \leq C \Xi^{\frac{1}{2}} \left(\int_0^t \|\zeta\|^2 d\tau \right)^{\frac{1}{4}} + C \|\zeta\| + \|\varrho\|. \quad (5.44)$$

Combining (5.44), (5.2) and (5.3), we obtain (5.42). \square

5.2. Error analysis for fully discrete scheme. In this subsection, we present some convergence results to the fully discrete scheme for the degenerate case and super convergence for the nondegenerate case.

Let $\bar{p}^n(\cdot) = \bar{p}(\cdot, t_n)$, $\mathbf{v}^n(\cdot) = \mathbf{v}(\cdot, t_n)$ and $\mathbf{u}^n(\cdot) = \mathbf{u}(\cdot, t_n)$ be the true solution evaluated at the discrete time levels. We will also denote $\pi p^n \in W_h$, $\pi \mathbf{s}^n \in \tilde{W}_h$ and $\Pi \mathbf{u}^n \in V_h$ to be the projections of the true solutions at the discrete time levels.

We rewrite (3.3) with $t = t_n$. Using the definitions of projections and assumption that $\nabla \cdot V_h \subset W_h$, standard manipulations show that the true solution satisfies the discrete equation

$$\left(\frac{\pi \bar{p}^n - \pi \bar{p}^{n-1}}{\Delta t}, w_h \right) + (\nabla \cdot \Pi \mathbf{u}^n, w_h) = (f^n - \Psi_t^n, w_h) + (\epsilon^n, w_h), \quad (5.45a)$$

$$(\Pi \mathbf{u}^n, \mathbf{z}_h) + (K(|\mathbf{s}^n|) \mathbf{s}^n, \mathbf{z}_h) = 0, \quad (5.45b)$$

$$(\pi \mathbf{s}^n, \mathbf{v}_h) + (\pi \bar{p}^n, \nabla \cdot \mathbf{v}_h) = (\nabla \Psi^n, \mathbf{v}_h), \quad (5.45c)$$

where ϵ^n is the time truncation error of order Δt .

THEOREM 5.9. Assume $(\bar{p}^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(\bar{p}_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h^n, \mathbf{u}_h^n, \mathbf{s}_h^n)$ solve the fully discrete mixed finite element approximation (3.8) for each time step n , $n = 1 \dots, N$. There exists a positive constant C independent of h and Δt such that if the Δt is sufficiently small then

$$\|\bar{p}_h^m - \bar{p}^m\| \leq C \left(\Upsilon \sum_{n=1}^m \Delta t \|\zeta^n\|_{L^\beta(\Omega)}^2 \right)^{\frac{1}{2}} + C \|\theta^m\| + C \Delta t \quad (5.46)$$

for all $m = 1, \dots, N$.

Consequently, if $p^n, \Psi^n \in H^{r+1}(\Omega)$ and $\mathbf{s}^n \in (W^{r+1, \beta}(\Omega))^d$ for $n = 1, \dots, N$ then for m between 1 and N ,

$$\|p_h^m - p^m\| \leq C(h^{r+1} + \Delta t). \quad (5.47)$$

Proof. Subtract (5.45) from (3.8), in the resultants using $w_h = \vartheta^n$, $\mathbf{z}_h = \eta^n$, $\mathbf{v}_h = \rho^n$ we obtain the error equations:

$$\left(\frac{\vartheta^n - \vartheta^{n-1}}{\Delta t}, \vartheta^n \right) + (\nabla \cdot \rho^n, \vartheta^n) = (\epsilon^n, \vartheta^n), \quad (5.48a)$$

$$(\rho^n, \eta^n) + (K(|\mathbf{s}_h^n|) \mathbf{s}_h^n - K(|\mathbf{s}^n|) \mathbf{s}^n, \eta^n) = 0, \quad (5.48b)$$

$$(\eta^n, \rho^n) + (\vartheta^n, \nabla \cdot \rho^n) = 0. \quad (5.48c)$$

Combining (5.48a)–(5.48c) gives

$$\|\vartheta^n\|^2 + \Delta t (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \eta^n) = (\vartheta^n, \vartheta^{n-1}) + \Delta t(\epsilon^n, \vartheta^n).$$

We rewrite this equation as form

$$\begin{aligned} & \|\vartheta^n\|^2 + \Delta t (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \mathbf{s}_h^n - \mathbf{s}^n) \\ &= (\vartheta^n, \vartheta^{n-1}) + \Delta t \left\{ (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \zeta^n) + (\epsilon^n, \vartheta^n) \right\}. \end{aligned} \quad (5.49)$$

The second term of (5.49), using (2.18), gives

$$(K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \mathbf{s}_h^n - \mathbf{s}^n) \geq C\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2. \quad (5.50)$$

where $\omega^n = \omega(t_n)$ defined as in (5.10).

The right hand side of (5.49), using Young's inequality, (5.11), (5.12) and (5.13), gives

$$\begin{aligned} & (\vartheta^n, \vartheta^{n-1}) + \Delta t ((K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \zeta^n) + (\epsilon^n, \vartheta^n)) \\ & \leq \frac{1}{2} (\|\vartheta^n\|^2 + \|\vartheta^{n-1}\|^2) + \Delta t \left\{ C\Upsilon \|\zeta^n\|_{L^\beta(\Omega)}^2 + \frac{1}{2} (\|\vartheta^n\|^2 + \|\epsilon^n\|^2) \right\}. \end{aligned} \quad (5.51)$$

From (5.49), (5.50) and (5.51), we obtain

$$\begin{aligned} & \|\vartheta^n\|^2 - \|\vartheta^{n-1}\|^2 + C\Delta t\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \\ & \leq \Delta t \|\vartheta^n\|^2 + C\Delta t (\Upsilon \|\zeta^n\|_{L^\beta(\Omega)}^2 + \|\epsilon^n\|^2). \end{aligned}$$

Summing over n gives

$$\begin{aligned} & (1 - \Delta t) \|\vartheta^m\|^2 + C \sum_{n=1}^m \Delta t\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \\ & \leq \sum_{n=1}^{m-1} \Delta t \|\vartheta^n\|^2 + C \sum_{n=1}^m \Delta t (\Upsilon \|\zeta^n\|_{L^\beta(\Omega)}^2 + \|\epsilon^n\|^2). \end{aligned}$$

By discrete Gronwall's inequality,

$$\|\vartheta^m\|^2 + C \sum_{n=1}^m \Delta t\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C \sum_{n=1}^m \Delta t (\Upsilon \|\zeta^n\|_{L^\beta(\Omega)}^2 + \|\epsilon^n\|^2).$$

Therefore

$$\begin{aligned} & \|\bar{p}_h^m - \bar{p}^m\|^2 + \sum_{n=1}^m \Delta t\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \\ & \leq C\Upsilon \sum_{n=1}^m \Delta t \|\zeta^n\|_{L^\beta(\Omega)}^2 + \|\theta^m\|^2 + C(\Delta t)^2. \end{aligned}$$

This implies (5.46).

From (5.46) and triangle inequality we find that

$$\|p_h^m - p^m\| \leq C \left(\Upsilon \sum_{n=1}^m \Delta t \|\zeta^n\|_{L^\beta(\Omega)}^2 \right)^{\frac{1}{2}} + C \|\theta^m\| + C\Delta t + \|\pi\Psi^m - \Psi^m\|. \quad (5.52)$$

The project properties and (5.52) imply (5.47). \square

THEOREM 5.10. *Under assumptions of Theorem 5.9. If $\mathbf{s}^n \in (W^{r+1,2}(\Omega))^d$ for $n = 1, \dots, N$ then there is positive constant C independent of h and time step such that if Δt sufficiently small then for m between 1 and N ,*

$$\|\mathbf{s}_h^m - \mathbf{s}^m\|_{L^\beta(\Omega)} + \|\mathbf{u}_h^m - \mathbf{u}^m\|_{L^\beta(\Omega)} \leq C(h^{\frac{r+1}{2}} + \sqrt{\Delta t}). \quad (5.53)$$

Proof. Recall that the true solution satisfies the discrete equations

$$(p_t^n, w_h) + (\nabla \cdot \Pi \mathbf{u}^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h \quad (5.54a)$$

$$(\Pi \mathbf{u}^n, \mathbf{z}_h) + (K(|\mathbf{s}^n|)\mathbf{s}^n, \mathbf{z}_h) = 0, \quad \forall \mathbf{z}_h \in \tilde{W}_h, \quad (5.54b)$$

$$(\pi \mathbf{s}^n, \mathbf{v}_h) + (\pi p^n, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h, \quad (5.54c)$$

Subtracting (3.8) from (5.54), choosing $w_h = \vartheta^n$, $\mathbf{z}_h = \eta^n$, $\mathbf{v}_h = \rho^n$, we obtain

$$\left(\frac{p_h^n - p_h^{n-1}}{\Delta t} - p_t^n, \vartheta^n \right) + (\nabla \cdot \rho^n, \vartheta^n) = 0, \quad (5.55a)$$

$$(\rho^n, \eta^n) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \eta^n) = 0, \quad (5.55b)$$

$$(\eta^n, \rho^n) + (\vartheta^n, \nabla \cdot \rho^n) = 0. \quad (5.55c)$$

Above equations yield

$$\left(\frac{p_h^n - p_h^{n-1}}{\Delta t} - p_t^n, \vartheta^n \right) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \eta^n) = 0. \quad (5.56)$$

We use (5.9), (5.56) to find that

$$\begin{aligned} \omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 &\leq (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \mathbf{s}_h^n - \mathbf{s}^n) \\ &= (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \eta^n) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \zeta^n) \\ &= - \left(\frac{p_h^n - p_h^{n-1}}{\Delta t} - p_t^n, \vartheta^n \right) + (K(|\mathbf{s}_h^n|)\mathbf{s}_h^n - K(|\mathbf{s}^n|)\mathbf{s}^n, \zeta^n). \end{aligned} \quad (5.57)$$

Due to (5.11), Cauchy-Schwartz and triangle inequality, one has

$$\omega^n \|\mathbf{s}_h^n - \mathbf{s}^n\|_{L^\beta(\Omega)}^2 \leq C((\Delta t)^{-1} \|p_h^n - p_h^{n-1}\| + \|p_t^n\|) \|\vartheta^n\| + \Upsilon \|\zeta^n\|_{L^\beta(\Omega)}.$$

Since

$$\begin{aligned} (\Delta t)^{-1} \|\bar{p}_h^m - p_h^{m-1}\| &= (\Delta t)^{-1} \left\| \int_{t_{m-1}}^{t_m} p_{h,t} dt \right\| \\ &\leq (\Delta t)^{-1} \int_{t_{m-1}}^{t_m} \|\bar{p}_{h,t}\| dt \\ &\leq \sup_{[T/N, T]} \|\bar{p}_{h,t}\| \leq \Xi, \end{aligned}$$

and also note that $\|\bar{p}_t^m\| \leq \sup_{[T/N, T]} \|\bar{p}_t\| \leq \Xi$, we have

$$\begin{aligned} \omega^n \|\mathbf{s}_h^m - \mathbf{s}^m\|_{L^\beta(\Omega)}^2 &\leq C\Xi \|\vartheta^m\| + \Upsilon \|\zeta^m\|_{L^\beta(\Omega)}^2 \\ &\leq C\Xi \left[\sum_{n=1}^m \Delta t \left(\Upsilon \|\zeta^n\|_{L^\beta(\Omega)}^2 + \|\epsilon^n\|^2 \right) \right]^{\frac{1}{2}} + \Upsilon \|\zeta^m\|_{L^\beta(\Omega)}^2. \end{aligned}$$

Using (5.31), we obtain

$$\|\mathbf{s}_h^m - \mathbf{s}^m\|_{L^\beta(\Omega)}^2 \leq C\Upsilon^\gamma \Xi \left[\left(\Upsilon \sum_{n=1}^m \Delta t \|\zeta^n\|_{L^\beta(\Omega)}^2 \right)^{\frac{1}{2}} + \Delta t \right] + C\Upsilon^{\gamma+1} \|\zeta^m\|_{L^\beta(\Omega)}^2.$$

It follows that

$$\begin{aligned} \|\mathbf{s}_h^m - \mathbf{s}^m\|_{L^\beta(\Omega)} &\leq C\Upsilon^{\frac{2\gamma+1}{4}} \Xi^{\frac{1}{2}} h^{\frac{r+1}{2}} \left(\sum_{n=1}^m \Delta t \|\mathbf{s}^n\|_{r+1,\beta}^2 d\tau \right)^{\frac{1}{2}} \\ &\quad + C\Upsilon^{\frac{\gamma+1}{2}} h^{r+1} \|\mathbf{s}^m\|_{r+1,\beta} + C\Xi\Upsilon^\gamma (\Delta t)^{\frac{1}{2}} \\ &\leq C(h^{\frac{r+1}{2}} + \sqrt{\Delta t}). \end{aligned} \quad (5.58)$$

Now we subtract (5.45b) from (3.8b), use $\mathbf{z}_h = (\rho^m)^{\beta-1}$, we have

$$(\rho^m, (\rho^m)^{\beta-1}) + (K(|\mathbf{s}_h^m|)\mathbf{s}_h^m - K(|\mathbf{s}^m|)\mathbf{s}^m, (\rho^m)^{\beta-1}) = 0.$$

In a similar way as in Theorem 5.5 we find that

$$\|\rho^m\|_{L^\beta(\Omega)} \leq C \|\mathbf{s}^m - \mathbf{s}_h^m\|_{L^\beta(\Omega)}.$$

which implies

$$\|\mathbf{u}_h^m - \mathbf{u}^m\|_{L^\beta(\Omega)} \leq C(\|\mathbf{s}^m - \mathbf{s}_h^m\|_{L^\beta(\Omega)} + \|\varrho^m\|_{L^\beta(\Omega)}) \leq C(h^{\frac{r+1}{2}} + \sqrt{\Delta t}). \quad (5.59)$$

Thus (5.58) and (5.59) lead to (5.53). The proof is complete. \square Finally we derives L^2 -estimates for $p_h^m - p^m$, $\mathbf{s}_h^m - \mathbf{s}^m$ and $\mathbf{u}_h^m - \mathbf{u}^m$ in the nondegenerate case. As results of Theorem 5.6 and 5.8 we can obtain the following error estimates

THEOREM 5.11. *Suppose $(\bar{p}^0, \mathbf{u}^0, \mathbf{s}^0) \in W \times V \times \tilde{W}$ and $(\bar{p}_h^0, \mathbf{u}_h^0, \mathbf{s}_h^0) \in W_h \times V_h \times \tilde{W}_h$. Let $(p, \mathbf{u}, \mathbf{s})$ solve problem (3.3) and $(p_h^n, \mathbf{u}_h^n, \mathbf{s}_h^n)$ solve the fully discrete mixed finite element approximation (3.8) for each time step n . Assume $p^n, \Psi^n \in H^{r+1}(\Omega)$ and $\mathbf{s}^n \in (L^\infty(\Omega) \cap H^{r+1}(\Omega))^d$, for $n = 1, \dots, N$. There exists a positive constant C independent of h such that if Δt sufficiently small then for m between 1 and N ,*

(i)

$$\begin{aligned} \|p_h^m - p^m\| + \left(\sum_{n=1}^m \Delta t \|\mathbf{s}_h^n - \mathbf{s}^n\|^2 \right)^{\frac{1}{2}} \\ + \left(\sum_{n=1}^m \Delta t \|\mathbf{u}_h^n - \mathbf{u}^n\|^2 \right)^{\frac{1}{2}} \leq C(h^{r+1} + \Delta t). \end{aligned} \quad (5.60)$$

(ii)

$$\|\mathbf{s}_h^m - \mathbf{s}^m\| + \|\mathbf{u}_h^m - \mathbf{u}^m\| \leq C(h^{\frac{r+1}{2}} + \sqrt{\Delta t}). \quad (5.61)$$

6. Numerical results. In this section, we give a simple numerical result illustrating the convergence theory. We test the convergence of our method with the Forchheimer two term law. For simplicity, consider $g(s) = 1 + s, s \geq 0$. Equation (2.3) shows that $s = \frac{-1 + \sqrt{1 + 4\xi}}{2}$ and

$$K(\xi) = \frac{1}{g(s(\xi))} = \frac{2}{1 + \sqrt{1 + 4\xi}}.$$

Since we analyze a first order time discretization, we consider the lowest order mixed method. Here we use the lowest order Raviart-Thomas mixed finite element on the unit square in two dimensions. Let $x = (x_1, x_2), \Omega = [0, 1]^2$. The analytical solution is chosen by

$$\begin{aligned} p(x, t) &= tx_1(1 - x_1)x_2(1 - x_2), \\ \mathbf{s}(x, t) &= t((1 - 2x_1)x_2(1 - x_2), x_1(1 - x_1)(1 - 2x_2)), \\ \mathbf{u}(x, t) &= \frac{2\mathbf{s}(x, t)}{1 + \sqrt{1 + 4|\mathbf{s}(x, t)|}} \end{aligned}$$

for all $x \in \Omega, t \in [0, 1]$. The forcing term f is determined accordingly to the analytical solution by $p_t - \nabla \cdot \mathbf{u} = f, (x, t) \in \Omega \times [0, 1]$. The initial data $p(x, 0) = 0$ and boundary data $p(x, t) = 0$ for all $(x, t) \in \partial\Omega \times [0, 1]$.

We divided the unit square into an $N \times N$ mesh of squares, each then subdivided into two right triangles. For each mesh, we solved the generalized Forchheimer equation numerically. The error control in each nonlinear solve is $\varepsilon = 10^{-6}$. Our problem is solved at each time level start at $t = 0$ until final time $t = 1$. At this time, we measured the L^2 -errors of pressure and L^β -errors of gradient of pressure and flux with $\beta = 2 - a = 2 - \frac{\deg(g)}{\deg(g)+1} = \frac{3}{2}$. The numerical results are listed as the following table.

N	$\ p - p_h\ $	Rates	$\ \mathbf{s} - \mathbf{s}_h\ _{L^\beta(\Omega)}$	Rates	$\ \mathbf{u} - \mathbf{u}_h\ _{L^\beta(\Omega)}$	Rates
4	0.00070985	-	0.052427	-	0.0492867	-
8	0.000307278	2.31	0.0277362	1.89	0.0267065	1.85
16	0.000142125	2.16	0.016163	1.72	0.0158135	1.69
32	7.44E-05	1.91	0.0115456	1.4	0.0113782	1.39
64	3.83E-05	1.94	0.01003	1.15	0.00990809	1.15
128	1.93E-05	1.99	0.00959868	1.04	0.00948783	1.04
256	9.65E-06	2.00	0.00948485	1.01	0.00937671	1.01
512	4.83E-06	2.00	0.0094558	1.00	0.00934833	1.00

Table 1. Convergence study for generalized Forchheimer equation in 2D.

Appendix A. We state the parabolic embedding and fast decaying geometry sequences lemmas which are used in our proof of Theorem 4.5. Let us denote throughout $Q_T = \Omega \times (0, T)$.

LEMMA A.1 (cf. [18]). Assume $\mu \geq 2$ and $\mu > \frac{ad}{\beta}$. Let

$$q = \mu \left(1 + \frac{\beta}{d} \right) - a.$$

Then

$$\|u\|_{L^q(Q_T)} \leq C(1 + \delta T)^{1/q} \|u\|,$$

where $\delta = 1$ in general, $\delta = 0$ in case u vanishes on the boundary $\partial\Omega$, and

$$[[u]] = \max_{[0,T]} \|u(t)\|_{L^\mu(\Omega)} + \left(\int_0^T \int_\Omega |u|^{\mu-2} |\nabla u|^\beta dx dt \right)^{\frac{1}{\mu-a}}.$$

LEMMA A.2 (cf. [19]). Let $\{Y_i\}_{i=0}^\infty$ be a sequence of non-negative numbers satisfying

$$Y_{i+1} \leq \sum_{k=1}^m A_k B_k^i Y_i^{1+\mu_k}, \quad i = 0, 1, 2, \dots,$$

where $A_k > 0$, $B_k > 1$ and $\mu_k > 0$ for $k = 1, 2, \dots, m$. Let $B = \max\{B_k : 1 \leq k \leq m\}$ and $\mu = \min\{\mu_k : 1 \leq k \leq m\}$. Then the following statements hold true.

$$\text{If } \sum_{k=1}^m A_k Y_0^{\mu_k} \leq B^{-1/\mu} \quad \text{then } \lim_{i \rightarrow \infty} Y_i = 0.$$

In particular,

$$\text{if } Y_0 \leq \min\{(m^{-1} A_k^{-1} B^{-\frac{1}{\mu}})^{1/\mu_k} : 1 \leq k \leq m\} \quad \text{then } \lim_{i \rightarrow \infty} Y_i = 0.$$

Proof. Proof of Theorem 4.5. We follow De Giorgi's technique (see [23]). First, we rewrite (3.2) as

$$\bar{p}_t - \nabla \cdot (K(|\nabla p|) \nabla p) = f - \Psi_t. \quad (\text{A.1})$$

For any $k \geq 0$, let

$$\bar{p}^{(k)} = \max\{\bar{p} - k, 0\}, \quad S_k(t) = \{x \in \Omega : \bar{p}^{(k)}(x, t) \geq 0\}, \quad \sigma_k = \int_0^T |S_k(t)| dt.$$

Let $k \geq \|\bar{p}_0\|_{L^\infty}$. Then $\bar{p}^{(k)}(x, 0) = 0$. Multiplying (A.1) by $|\bar{p}^{(k)}|^{\mu-1}$ and integrating over the domain Ω give

$$\begin{aligned} \frac{1}{\mu} \frac{d}{dt} \int_\Omega |\bar{p}^{(k)}|^\mu dx + (\mu - 1) \int_\Omega |\bar{p}^{(k)}|^{\mu-2} K(|\nabla p|) \nabla p \cdot \nabla \bar{p}^{(k)} dx \\ = \int_\Omega (f - \Psi_t) |\bar{p}^{(k)}|^{\mu-1} dx. \end{aligned} \quad (\text{A.2})$$

Since $\nabla \bar{p}^{(k)} = \nabla \bar{p} = \nabla p - \nabla \Psi$, (A.2) implies

$$\begin{aligned} \frac{d}{dt} \int_\Omega |\bar{p}^{(k)}|^\mu dx + \int_\Omega |\bar{p}^{(k)}|^{\mu-2} K(|\nabla p|) |\nabla p|^2 dx \\ \leq C \int_\Omega |f - \Psi_t| |\bar{p}^{(k)}|^{\mu-1} dx + C \int_\Omega |\bar{p}^{(k)}|^{\mu-2} K(|\nabla p|) |\nabla p| |\nabla \Psi| dx. \end{aligned} \quad (\text{A.3})$$

Using (2.11), we have

$$\begin{aligned} |\bar{p}^{(k)}|^{\mu-2} K(|\nabla p|) |\nabla p|^2 &\geq C \left(|\bar{p}^{(k)}|^{\mu-2} |\nabla p|^\beta - |\bar{p}^{(k)}|^{\mu-2} \right) \\ &\geq C \left(|\bar{p}^{(k)}|^{\mu-2} |\nabla \bar{p}^{(k)}|^\beta - |\bar{p}^{(k)}|^{\mu-2} |\nabla \Psi|^\beta - |\bar{p}^{(k)}|^{\mu-2} \right). \end{aligned} \quad (\text{A.4})$$

Also

$$\begin{aligned} |\bar{p}^{(k)}|^{\mu-2} K(|\nabla p|)|\nabla p||\nabla \Psi| &\leq |\bar{p}^{(k)}|^{\mu-2} (|\nabla \bar{p}|^{\beta-1} + |\nabla \Psi|^{\beta-1}) |\nabla \Psi| \\ &\leq |\bar{p}^{(k)}|^{\mu-2} |\nabla \bar{p}|^{\beta-1} |\nabla \Psi| + |\bar{p}^{(k)}|^{\mu-2} |\nabla \Psi|^\beta. \end{aligned}$$

Young's inequality provides

$$|\nabla \bar{p}|^{\beta-1} |\nabla \Psi| \leq \varepsilon |\nabla \bar{p}^{(k)}|^\beta + C\varepsilon^{1-\beta} |\nabla \Psi|^\beta.$$

Thus

$$\begin{aligned} |\bar{p}^{(k)}|^{\mu-2} K(|\nabla p|)|\nabla p||\nabla \Psi| \\ \leq \varepsilon |\bar{p}^{(k)}|^{\mu-2} |\nabla \bar{p}^{(k)}|^\beta + C(1 + \varepsilon^{1-\beta}) |\bar{p}^{(k)}|^{\mu-2} |\nabla \Psi|^\beta. \end{aligned} \quad (\text{A.5})$$

Combining (A.3), (A.4) and (A.5), selecting $\varepsilon = C/2$ we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\bar{p}^{(k)}|^\mu dx + \int_{\Omega} |\bar{p}^{(k)}|^{\mu-2} |\nabla \bar{p}^{(k)}|^\beta dx \\ \leq C \int_{\Omega} |f - \Psi_t| |\bar{p}^{(k)}|^{\mu-1} dx + C \int_{\Omega} |\bar{p}^{(k)}|^{\mu-2} (|\nabla \Psi|^\beta + 1) dx \\ \leq \delta \int_{\Omega} |\bar{p}^{(k)}|^\mu dx + C\delta^{1-\mu} \int_{\Omega} |f - \Psi_t|^\mu \chi_k dx \\ \quad + C\delta^{1-\frac{\mu}{2}} (1 + \|\nabla \Psi\|_{L^\infty})^{\frac{\beta\mu}{2}} |S_k(t)| \\ \leq \delta \int_{\Omega} |\bar{p}^{(k)}|^\mu dx + C\delta^{1-\mu} \|f - \Psi_t\|_{L^{\mu+1}}^\mu |S_k(t)| \\ \quad + C\delta^{1-\frac{\mu}{2}} (1 + \|\nabla \Psi\|_{L^\infty})^{\frac{\beta\mu}{2}} |S_k(t)|. \end{aligned}$$

Here $\chi_k(t)$ is the characteristics function of $S_k(t)$. In previous inequality integrating from 0 to T and selecting $\delta = 1/(2T)$ we find that

$$\begin{aligned} \sup_{[0,T]} \int_{\Omega} |\bar{p}^{(k)}|^\mu dx + \int_0^T \int_{\Omega} |\bar{p}^{(k)}|^{\mu-2} |\nabla \bar{p}^{(k)}|^\beta dx dt \\ \leq C \int_0^T \left(T^{\mu-1} \|f - \Psi_t\|_{L^{\mu+1}}^\mu + T^{\frac{\mu}{2}-1} (1 + \|\nabla \Psi\|_{L^\infty}^{\frac{\beta\mu}{2}}) \right) |S_k(t)| dt. \end{aligned}$$

Let

$$\begin{aligned} F_k &= \sup_{[0,T]} \int_{\Omega} |\bar{p}^{(k)}|^\mu dx + \int_0^T \int_{\Omega} |\bar{p}^{(k)}|^{\mu-2} |\nabla \bar{p}^{(k)}|^\beta dx dt, \quad \sigma_k = \int_0^T |S_k(t)| dt, \\ \mathcal{E}_T &= T^{\mu-1} \|f - \Psi_t\|_{L_t^\infty(0,T;L_x^{\mu+1})}^\mu + T^{\frac{\mu}{2}-1} (1 + \|\nabla \Psi\|_{L_t^\infty(0,T;L_x^\infty)}^{\frac{\beta\mu}{2}}). \end{aligned} \quad (\text{A.6})$$

Then $F_k \leq C\mathcal{E}_T\sigma_k$. Let $\mu > 0$ be sufficient large, q as in (A.1). By Lemma A.1:

$$\|\bar{p}^{(k)}\|_{L^q(Q_T)} \leq C(1 + T)^{1/q} (F_k^{1/\mu} + F_k^{1/(\mu-a)}). \quad (\text{A.7})$$

Let $k_i = M_0(2 - 2^{-i})$, for $i = 0, 1, 2, \dots$, then k_i is increasing in i , S_{k_i} and σ_{k_i} are decreasing.

Note that we want $k_0 = M_0 \geq \|\bar{p}_0\|_{L^\infty}$. By definition,

$$\|\bar{p}^{(k_i)}\|_{L^q(Q_T)} \geq \|\bar{p}^{(k_i)}\|_{L^q(Q_{k_{i+1}})} \geq (k_{i+1} - k_i) \sigma_{k_{i+1}}^{1/q}, \quad (\text{A.8})$$

where $\mathcal{Q}_k = \{(x, t) \in U \times (0, T) : p(x, t) > k\}$.

Combining (A.8) and (A.7), we obtain

$$\sigma_{k_{i+1}}^{1/q} \leq \frac{C(1+T)^{1/q}}{k_{i+1} - k_i} \left[F_{k_i}^{1/\mu} + F_{k_i}^{1/(\mu-a)} \right].$$

Hence

$$\sigma_{k_{i+1}} \leq C(1+T) \frac{2^{qi}}{M_0^q} \left[(\mathcal{E}_T \sigma_{k_i})^{q/\mu} + (\mathcal{E}_T \sigma_{k_i})^{q/(\mu-a)} \right]. \quad (\text{A.9})$$

Let

$$Y_i = \sigma_{k_i}, \quad A_1 = C(1+T) \mathcal{E}_T^{q/\mu} M_0^{-q}, \quad A_2 = C(1+T) \mathcal{E}_T^{q/(\mu-a)} M_0^{-q}, \quad B = 2^q,$$

$$\mu_0 = q/\mu - 1, \quad \nu_0 = q/(\mu - a) - 1.$$

Then (A.9) rewrites as

$$Y_{i+1} \leq B(A_1 Y_i^{1+\mu_0} + A_2 Y_i^{1+\nu_0}).$$

Note that $\mu_0 < \nu_0$, $k_0 = M_0 \geq \|\bar{p}_0\|_{L^\infty}$, $Y_0 = \sigma_{M_0} \leq |Q_T| = CT$.

Choose M_0 large such that

$$T + 1 \leq C \min \left\{ A_1^{-1/\mu_0}, A_2^{-1/\nu_0} \right\}. \quad (\text{A.10})$$

Explicitly,

$$\begin{aligned} M_0 &\geq C(1+T)^{\frac{\mu_0+1}{q}} \mathcal{E}_T^{\frac{1}{\mu}} = C(1+T)^{\frac{1}{\mu}} \mathcal{E}_T^{\frac{1}{\mu}}, \\ M_0 &\geq (1+T)^{\frac{\nu_0+1}{q}} \mathcal{E}_T^{\frac{1}{\mu-a}} = (1+T)^{\frac{1}{\mu-a}} \mathcal{E}_T^{\frac{1}{\mu-a}}. \end{aligned}$$

Since

$$\begin{aligned} (1+T) \mathcal{E}_T &\leq C(1+T)^\mu \|f - \Psi_t\|_{L_t^\infty(0, T; L_x^{\mu+1})}^\mu \\ &\quad + C(1+T)^{\frac{\mu}{2}} (1 + \|\nabla \Psi\|_{L_t^\infty(0, T; L_x^\infty)}^\beta)^{\frac{\mu}{2}}. \end{aligned}$$

We select

$$\begin{aligned} M_0 &= \|\bar{p}_0\|_{L^\infty} + C \left\{ (1+T)^\mu (1 + \|f - \Psi_t\|_{L_t^\infty(0, T; L_x^{\mu+1})})^\mu \right. \\ &\quad \left. + (1+T)^{\frac{\mu}{2}} (1 + \|\nabla \Psi\|_{L_t^\infty(0, T; L_x^\infty)}^\beta)^{\frac{\mu}{2}} \right\}^{\frac{1}{\mu-a}}. \end{aligned}$$

Then (A.10) holds. Applying A.2 with $m = 2$, we have

$$\sigma_{2M_0} = \lim_{i \rightarrow \infty} \sigma_{k_i} = 0,$$

that is

$$\bar{p}(x, t) \leq 2M_0 \quad \text{a.e. in } Q_T.$$

Replacing p by $-p$, ψ by $-\psi$. We finish the proof. \square

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REFERENCES

- [1] TODD ARBOGAST, MARY F. WHEELER, AND IVAN YOTOV, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.
- [2] TODD ARBOGAST, MARY F. WHEELER, AND NAI-YING ZHANG, *A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media*, SIAM J. Numer. Anal., 33 (1996), pp. 1669–1687.
- [3] EUGENIO AULISA, LIDIA BLOSHANSKAYA, LUAN HOANG, AND AKIF IBRAGIMOV, *Analysis of generalized Forchheimer flows of compressible fluids in porous media*, J. Math. Phys., 50 (2009), pp. 103102, 44.
- [4] FRANCO BREZZI, JIM DOUGLAS, JR., AND L. D. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.
- [5] FRANCO BREZZI AND MICHEL FORTIN, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.
- [6] A. O. ÇELEBI, V. K. KALANTAROV, AND D. UĞURLU, *On continuous dependence on coefficients of the Brinkman-Forchheimer equations*, Appl. Math. Lett., 19 (2006), pp. 801–807.
- [7] J. CHADAM AND Y. QIN, *Spatial decay estimates for flow in a porous medium*, SIAM J. Math. Anal., 28 (1997), pp. 808–830.
- [8] PHILIPPE G. CIARLET, *The finite element method for elliptic problems*, North-Holland Publishing Co., Amsterdam, 1978. Studies in Mathematics and its Applications, Vol. 4.
- [9] CLINT N. DAWSON AND MARY F. WHEELER, *Two-grid methods for mixed finite element approximations of nonlinear parabolic equations*, in Domain decomposition methods in scientific and engineering computing (University Park, PA, 1993), vol. 180 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1994, pp. 191–203.
- [10] R. E. EWING, R. D. LAZAROV, J. E. PASCIAK, AND A. T. VASSILEV, *Mathematical modeling, numerical techniques, and computer simulation of flows and transport in porous media*, in Computational techniques and applications: CTAC95 (Melbourne, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 13–30.
- [11] P. FORCHHEIMER, *Wasserbewegung durch Boden Zeit*, vol. 45, Ver. Deut. Ing., 1901.
- [12] F. FRANCHI AND B. STRAUGHAN, *Continuous dependence and decay for the Forchheimer equations*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 459 (2003), pp. 3195–3202.
- [13] M. GENTILE AND B. STRAUGHAN, *Structural stability in resonant penetrative convection in a Forchheimer porous material*, Nonlinear Anal. Real World Appl., 14 (2013), pp. 397–401.
- [14] V. GIRAULT AND M. F. WHEELER, *Numerical discretization of a Darcy-Forchheimer model*, Numer. Math., 110 (2008), pp. 161–198.
- [15] LUAN HOANG AND AKIF IBRAGIMOV, *Structural stability of generalized Forchheimer equations for compressible fluids in porous media*, Nonlinearity, 24 (2011), pp. 1–41.
- [16] ———, *Qualitative study of generalized Forchheimer flows with the flux boundary condition*, Adv. Diff. Eq., 17 (2012), pp. 511–556.
- [17] LUAN T. HOANG, AKIF IBRAGIMOV, THINH T. KIEU, AND ZEEV SOBOL, *Stability of solutions to generalized Forchheimer equations of any degree*, (2012). Submitted.
- [18] LUAN T. HOANG AND THINH T. KIEU, *Interior estimates for generalized forchheimer flows of slightly compressible fluids*, (2014). Submitted.
- [19] LUAN T. HOANG, THINH T. KIEU, AND TUOC V. PHAN, *Properties of generalized Forchheimer flows in porous media*, Journal of Mathematical Sciences, 202 (2014), pp. 259–332.
- [20] DAVID JERISON AND CARLOS E. KENIG, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal., 130 (1995), pp. 161–219.
- [21] CLAES JOHNSON AND VIDAR THOMÉE, *Error estimates for some mixed finite element methods for parabolic type problems*, RAIRO Anal. Numér., 15 (1981), pp. 41–78.
- [22] M.-Y. KIM AND E.-J. PARK, *Fully discrete mixed finite element approximations for non-Darcy flows in porous media*, Comput. Math. Appl., 38 (1999), pp. 113–129.
- [23] O. A. LADYŽENSKAJA, V. A. SOLONNIKOV, AND N. N. URAL’CEVA, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968.
- [24] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.
- [25] CARLO MIRANDA, *Partial differential equations of elliptic type*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Springer-Verlag, New York-Berlin, 1970. Second revised edition. Translated from the Italian by Zane C. Motteler.
- [26] MORRIS MUSKAT, *The flow of homogeneous fluids through porous media*, McGraw-Hill Book

- Company, inc., 1937.
- [27] J.-C. NÉDÉLEC, *Mixed finite elements in \mathbf{R}^3* , Numer. Math., 35 (1980), pp. 315–341.
 - [28] HAO PAN AND HONGXING RUI, *Mixed element method for two-dimensional Darcy-Forchheimer model*, J. Sci. Comput., 52 (2012), pp. 563–587.
 - [29] EUN-JAE PARK, *Mixed finite element methods for generalized Forchheimer flow in porous media*, Numer. Methods Partial Differential Equations, 21 (2005), pp. 213–228.
 - [30] L. E. PAYNE AND B. STRAUGHAN, *Convergence and continuous dependence for the Brinkman-Forchheimer equations*, Stud. Appl. Math., 102 (1999), pp. 419–439.
 - [31] P. BASAK, *Non-darcy flow and its implications to seepage problems*, Journal of the Irrigation and Drainage Division, 103 (1977), pp. 459–473.
 - [32] Y. QIN AND P. N. KALONI, *Spatial decay estimates for plane flow in Brinkman-Forchheimer model*, Quart. Appl. Math., 56 (1998), pp. 71–87.
 - [33] P. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.
 - [34] HONGXING RUI AND HAO PAN, *A block-centered finite difference method for the Darcy-Forchheimer model*, SIAM J. Numer. Anal., 50 (2012), pp. 2612–2631.
 - [35] R. E. SHOWALTER, *Monotone operators in Banach space and nonlinear partial differential equations*, vol. 49 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997.
 - [36] CAROL S. WOODWARD AND CLINT N. DAWSON, *Analysis of expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media*, SIAM J. Numer. Anal., 37 (2000), pp. 701–724 (electronic).
 - [37] EBERHARD ZEIDLER, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.